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MATHEMATICS

8th YEAR

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PART 2

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NOTE TO THE TEACHER

Mathematics: 8th Year is presented in two parts of which this is the second. It contains Chapters VI through XI.

Chapters I through V were published in *Mathematics: 8th Year, Part 1.*

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MATHEMATICS

8th YEAR

PART 2

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FOREWORD

Because of the increasing dependence of our society upon trained mathematical manpower, it is essential that vital and contemporary mathematics be taught in our schools.

The mathematics program set forth in this publication has developed as a result of experimentation and evaluation in classroom situations. This is Part II of Mathematics: 8th Year. Part I, a separate bulletin, was published during the school year 1967-1968.

This bulletin represents a cooperative effort of the Bureau of Curriculum Development, the Bureau of Mathematics, and the Office of Junior High Schools.

We wish to thank the staff members who have so generously contributed to this work which should enhance the efforts of our teachers in developing pupil competency.

SEELIG LESTER
Deputy Superintendent of Schools

November, 1968

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CHAPTER VI

THE SET OF RATIONAL NUMBERS

The material in this chapter is concerned with developing concepts of negative rational numbers as an extension of the positive rational numbers and zero. Pupils are helped to see that the set of positive rational numbers and zero form a subset of the set of rational numbers, and that the set of integers is another important subset of the rational numbers.

Among the important understandings developed in this chapter are:

need for negative rational numbers
relationship between positive and negative rationals
 as shown on the number line
order in the system of rational numbers
operations in the set of rational numbers

The beginning lessons of the chapter investigate the need for negative rational numbers. In a previous grade, pupils have learned to represent the quotient of $1 \div 3$ as $\frac{1}{3}$; the quotient of $2 \div 5$ as $\frac{2}{5}$, etc. They know that when a whole number is divided by another whole number except zero, the resulting number is a rational number. But the quotient of $3 \div (-8)$ or $-\frac{3}{8}$ needs further explanation. Based on earlier work with integers, pupils are helped to understand the extension of the form $\frac{a}{b}$ where a and b represent integers $(b \neq 0)$.

The number line is used to visualize the concept of the additive inverse (opposite) of a rational number. Pupils should understand that the invention of negative rational numbers makes it possible for every rational number to have an additive inverse. The number line is also used to develop pupil understanding of the concept of order in the set of rationals.

As the operations of addition, subtraction, multiplication, and division of rational numbers are developed, pupils make use of their previous knowledge of whole numbers, non-negative rationals, and integers. It is important that pupils understand that the properties of the non-negative rational numbers hold for the set of rational numbers. Practice examples are included to explore these understandings.

The operations on rational numbers provide an excellent opportunity for reinforcement and practice in fundamental operations.

CHAPTER VI

THE SET OF RATIONAL NUMBERS

Lessons 54-60

Lesson 54

Topic: The Set of Rational Numbers

Aim: To discover a need for negative rational numbers

Specific Objectives:

To review the meaning of rational numbers in terms of whole numbers
To discover a need for negative rational numbers
To define rational numbers in terms of integers

Challenge: To what set of numbers does $-\frac{2}{5}$ belong?

I. Procedure

A. Meaning of a rational number

1. Have pupils recall that a rational number (of arithmetic) is one which can be expressed as a whole number divided by a whole number, other than zero.

2. Is $\frac{15}{2}$ a rational number? Why?

   Is $\frac{8}{4}$ a rational number? Why?

   Is 5 a rational number? Why?

   (Yes, because any whole number such as 5 can be expressed as a whole number divided by a whole number: $\frac{5}{1}$, $\frac{10}{2}$, $\frac{15}{3}$, etc.)

3. Have pupils conclude from the work above that every whole number is a rational number although not every rational number is a whole number. Therefore, the set of whole numbers is a subset of the set of rational numbers.

B. Need for negative rational numbers

1. Replace the frames to make these sentences true.

   $10 \div 5 = \square$ because $5 \times \square = 10$
   $-10 \div -2 = \square$ because $-2 \times \square = -10$
   $12 \div -3 = \square$ because $-3 \times \square = 12$
   $-15 \div -5 = \square$ because $-5 \times \square = -15$

   How were the signs of the quotient determined?

2. Consider $2 \div 5 = \square$.

   a. What is the sign of the quotient?

   b. What is the quotient?

   c. What kind of number is the quotient $\frac{2}{5}$? (positive rational)

3. Refer to challenge $2 \div -5 = \square$.

   a. According to the "rules" for division of integers, what must be the sign of the quotient?

   b. What is the quotient?

   c. What kind of number would you call $-\frac{2}{5}$?

4. Have pupils conclude that the set of rational numbers includes the negative rational numbers.

C. Redefining a rational number in terms of integers

1. Have pupils recall that if a number can be expressed as a whole number divided by a whole number other than zero, it is a positive rational number.

2. Lead the pupils to understand that this definition is related only to the non-negative rational numbers, since the set of whole numbers does not include negative numbers.

3. Elicit that since the set of rational numbers has been shown to include negative numbers, we may extend the definition of "rational number" as follows:

   If a number can be expressed as an integer divided by an integer other than zero, it is a rational number.

II. Practice

1. Complete the following:

   $-12 \div 2 = \square$ because $2 \times \square = -12$
   $3 \div 7 = \square$ because $7 \times \square = 3$
   $-3 \div -11 = \square$ because $-11 \times \square = -3$
   $-5 \div 8 = \square$ because $8 \times \square = -5$

2. Express in rational form:

   $165 \div -15 = ?$
   $-45 \div 15 = ?$
   $16 \div -12 = ?$
   $-13 \div 26 = ?$
   $-22 \div -23 = ?$

3. Compute the quotients:

$\frac{10}{5} = ?$

$\frac{5}{10} = ?$

$\frac{-9}{3} = ?$

$\frac{-3}{9} = ?$

$\frac{24}{-6} = ?$

$\frac{6}{-24} = ?$

$\frac{-45}{-3} = ?$

$\frac{-3}{-45} = ?$

III. Summary

A. How is the set of whole numbers related to the set of rational numbers?

B. If a positive integer is divided by a negative integer, the result is a _____ rational number.

C. If a negative integer is divided by a negative integer, the result is a _____ rational number.

D. How do we define a rational number in terms of integers?

Lesson 55

Topic: Rational Numbers

Aim: To develop by use of the number line, the concept of order in the set of rational numbers

Specific Objectives:

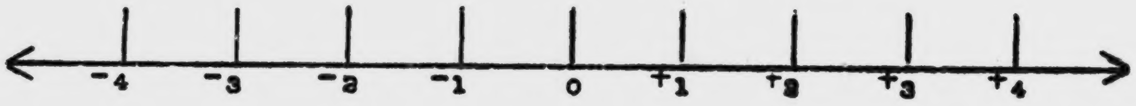
To show that each negative rational number can be associated with a point on the number line
To develop the concept of order in the set of rational numbers

Challenge: Indicate the opposite of $\frac{2}{3}$ on the number line.

I. Procedure

A. Negative rational numbers on the number line

1. Have pupils draw a number line as indicated below:



2. How many points are 3 units from zero? What numbers are associated with these points? (+3 and -3) Indicate these points on the number line.

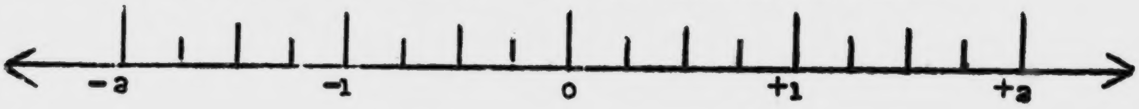
3. Why do you think that +3 and -3 are called opposites? (Because they are the same distance from zero, but in opposite directions.)

4. What is the opposite of $3\frac{1}{2}$, $+ 2\frac{1}{2}$, $-\frac{1}{2}$, $-2\frac{1}{2}$? Indicate these points on the number line.

5. Return to challenge. What is the opposite of $\frac{2}{3}$? Indicate this point on the number line.

B. Order in the set of rational numbers

1. Have pupils prepare a number line as indicated below:



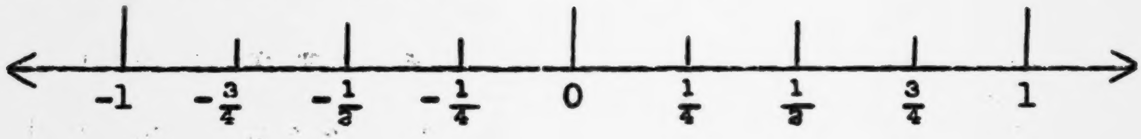
2. Which is greater 0 or 1; 0 or -1; -1 or 2, -1 or -2?

3. What can you say of the relative position of the points corresponding to 0 and 1; 0 and -1; -1 and 2; -1 and -2? (The greater number is associated with a point to the right of the point associated with the lesser number.)

4. Have pupils locate on a number line the points associated with the following pairs of numbers:

$\frac{3}{4}$ and $\frac{1}{4}$	$1\frac{1}{4}$ and $1\frac{1}{2}$
$-\frac{3}{4}$ and 0	$-\frac{1}{2}$ and $-\frac{1}{4}$

5. Consider the number line:



For each of the following pairs, insert the symbol > or < to make a true statement.

$\frac{1}{2}, \frac{1}{4}$	$-\frac{1}{2}, -\frac{1}{4}$
$-\frac{1}{4}, -\frac{1}{2}$	$-\frac{3}{4}, -\frac{1}{2}$

6. Elicit that the set of rational numbers follows the same pattern as the set of integers. The greater number is associated with a point to the right of the point associated with the lesser number.

II. Practice

1. What is the opposite of each of the following: 2, $\frac{1}{4}$, $-3\frac{1}{2}$, $-\frac{1}{3}$, 0?

2. Judge the following true or false.

$\frac{1}{4} > -\frac{1}{4}$	$0 < \frac{3}{4}$
$\frac{1}{4} < -\frac{1}{2}$	$0 > -1\frac{1}{2}$
$\frac{3}{4} > 0$	$-2 > 1\frac{3}{4}$

III. Summary

A. Explain by referring to the number line what we mean when we say -3 is the opposite of +3.

B. On the number line the point associated with the greater of two numbers is always to the _____ of the point associated with the lesser number.

C. How would you explain that on the number line the point associated with $\frac{9}{10}$ is to the right of the point associated with $\frac{1}{10}$, while the point associated with $-\frac{9}{10}$ is to the left of the point associated with $-\frac{1}{10}$.

Lesson 56

Topic: Addition and Subtraction of Non-negative Rational Numbers

Note to Teacher: This is essentially a review lesson. Use as indicated by the needs of the pupils.

Aim: To review addition and subtraction of non-negative rational numbers

Specific Objectives:

To review addition and subtraction of rational numbers named by fractions with like denominators
To review addition and subtraction of rational numbers named by fractions with unlike denominators

Challenge: In our family budget $\frac{1}{5}$ of the income is used for rent and $\frac{1}{4}$ is used for food. What fractional part of our income is used for both food and rent?

I. Procedure

A. Addition and subtraction of rational numbers with like denominators

1. Have pupils recall that if the denominators of the fractions are alike, we simply add (or subtract) the numerators to get the numerator of the sum (or difference).

2. Perform the indicated operations:

a. $\frac{1}{3} + \frac{1}{3}$ b. $\frac{2}{5} + \frac{2}{5}$ c. $\frac{2}{3} - \frac{1}{3}$ d. $\frac{3}{5} - \frac{1}{5}$

B. Addition and subtraction of rational numbers named by fractions with unlike denominators

1. Refer to challenge problem.

a. Elicit that to solve the problem, we must perform the following addition: $\frac{1}{5} + \frac{1}{4}$.

What must be true of the denominators before we can combine the numerators?

Elicit that each number must be renamed so that the denominators are alike. Have pupils suggest several names for $\frac{1}{5}$. List them on the board.

b. Consider the set of names for $\frac{1}{5}$.

$\{\frac{1}{5}, \frac{2}{10}, \frac{3}{15}, \frac{4}{20}, \frac{5}{25}, \frac{6}{30}, \frac{7}{35}, \frac{8}{40}, \ldots\}$

The set of denominators is $A = \{5, 10, 15, 20, 25, 30, 35, 40, \ldots\}$.

Elicit that A is also a set of multiples of 5, the denominator of $\frac{1}{5}$.

In like manner, elicit the set of names for $\frac{1}{4}$.

Then a set of multiples of 4, the denominator of $\frac{1}{4}$, is

$B = \{4, 8, 12, 16, 20, 24, 28, 32, 36, 40, \ldots\}$.

Recall that the common elements of sets A and B make up a set of common multiples of 5 and 4. $A \cap B = \{20, 40, 60, \ldots\}$.

c. Elicit that any member of the intersection set may be used as a common denominator to rename $\frac{1}{5}$ and $\frac{1}{4}$.

Lead pupils to see that the set of common multiples of the denominators of two fractions is the same as the set of common denominators.

Lead pupils to see the advisability of using the least common denominator, (L.C.D.), which, in the preceding illustration, is 20.

d. In the example $\frac{1}{5} + \frac{1}{4}$, a common denominator (a common multiple of the denominators) must contain 5 and 4 as factors. Since 5 and 4 have no factors in common other than 1, the product 5×4 or 20 is the least common denominator, (the least common multiple of the denominators.)

$$\begin{aligned}\frac{1}{5} + \frac{1}{4} &= \left(\frac{1}{5} \times \frac{4}{4}\right) + \left(\frac{1}{4} \times \frac{5}{5}\right)\\ &= \frac{4}{20} + \frac{5}{20}\\ &= \frac{9}{20}\end{aligned}$$

e. Consider $\frac{5}{8} - \frac{1}{3}$.

How would you suggest $\frac{5}{8}$ and $\frac{1}{3}$ be renamed before subtracting? (They should be renamed by fractions with like denominators.) What is the least common denominator of $\frac{5}{8}$ and $\frac{1}{3}$? (24)

How can $\frac{5}{8}$ and $\frac{1}{3}$ be renamed by fractions with 24 as the denominator?

$$\begin{aligned}\frac{5}{8} - \frac{1}{3} &= \left(\frac{5}{8} \times \frac{3}{3}\right) - \left(\frac{1}{3} \times \frac{8}{8}\right)\\ &= \frac{15}{24} - \frac{8}{24}\\ &= \frac{7}{24}\end{aligned}$$

II. Practice

A. Name a common denominator for each of the following pairs of fractions:

1. $\frac{1}{4}, \frac{3}{8}$ 2. $\frac{2}{5}, \frac{3}{10}$ 3. $\frac{1}{2}, \frac{1}{3}$ 4. $\frac{1}{2}, \frac{1}{5}$

5. $\frac{2}{3}, \frac{3}{4}$ 6. $\frac{7}{8}, \frac{2}{3}$ 7. $\frac{4}{5}, \frac{2}{3}$ 8. $\frac{1}{2}, \frac{2}{7}$

B. Add:

1. $\frac{1}{4} + \frac{3}{8}$ 2. $\frac{2}{5} + \frac{3}{10}$ 3. $\frac{1}{3} + \frac{1}{2}$ 4. $\frac{1}{2} + \frac{1}{5}$

5. $\frac{2}{3} + \frac{3}{4}$ 6. $\frac{7}{8} + \frac{2}{3}$ 7. $\frac{4}{5} + \frac{2}{3}$ 8. $\frac{1}{2} + \frac{2}{7}$

C. Subtract:

1 $\frac{5}{8} - \frac{1}{4}$ 2. $\frac{8}{9} - \frac{2}{3}$ 3. $\frac{1}{3} - \frac{1}{8}$ 4. $\frac{7}{8} - \frac{2}{3}$ 5. $\frac{4}{5} - \frac{2}{3}$

D. Add:

1. $\frac{1}{6} + \frac{2}{3} + \frac{1}{2}$ 2. $\frac{3}{5} + \frac{2}{3} + \frac{1}{6}$ 3. $3\frac{1}{2} + 2\frac{1}{3} + \frac{1}{6}$

III. Summary

A. How do we add or subtract two rational numbers that are named by fractions whose denominators are alike?

B. How do we add or subtract two rational numbers that are named by fractions whose denominators are not alike? (Rename the rational numbers by fractions with a common denominator.)

C. How do we find a common denominator for two or more unlike denominators?

D. Why is it an advantage to use the least common denominator in adding or subtracting fractions with unlike denominators?

Lesson 57

Topic: Addition of Positive and Negative Rational Numbers

Aim: To develop rules for the addition of positive and negative rational numbers

Specific Objectives:

To review addition in the set of integers
To apply the rules of addition in the set of integers to addition of rational numbers

Challenge: What is the sum of $\frac{5}{8} + (-\frac{1}{4})$?

I. Procedure

A. Review of addition in the set of integers

Note: Review, if necessary, the concept of absolute value as presented in the chapter on integers.

1. Consider the following examples:

$6 + 11 = ?$ $\qquad$ $5 + (-4) = ?$

$-7 + (-9) = ?$ $\qquad$ $-6 + 5 = ?$

2. Have pupils recall that:

a. When both addends are positive, their sum is positive. When both addends are negative, their sum is negative. In each case, the absolute value of the sum is the sum of the absolute values of each of the addends.

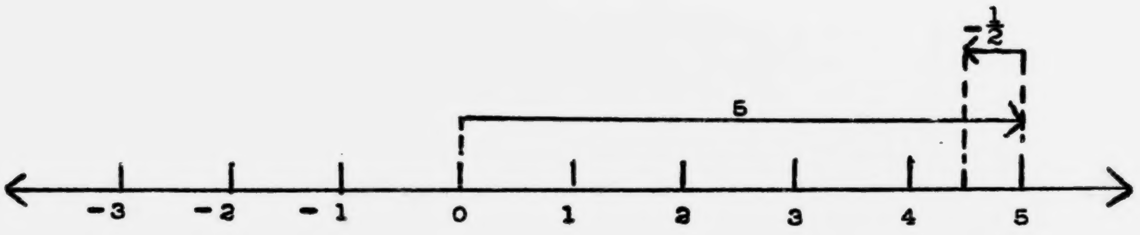
b. When one addend is positive and the other negative, the sign of the sum is determined by the sign of the addend with the greater absolute value. In this case, the absolute value of the sum is the difference between the absolute values of the two addends.

3. Find the sums of the examples in A-1.

B. Addition of positive and negative rational numbers

1. On the number line find the sum of $5 + (-\frac{1}{2})$.

Have pupils recall that on the number line addition of a positive number is represented by moving to the right and addition of a negative number by moving to the left.



Therefore, $5 + (-\frac{1}{2}) = 4\frac{1}{2}$.

2. Using the number line, find the following sums:

   a. $1\frac{1}{2} + \frac{1}{2}$

   b. $-4 + (-\frac{3}{4})$

   c. $\frac{3}{4} + (-\frac{1}{4})$

   d. $-1\frac{1}{2} + \frac{1}{2}$

3. What is the sign of the sum in example a above?
   What is the sign of the sum in example b above?

   What is the relationship of the absolute value of the sum to the absolute values of the addends in each of these examples?

4. What is the sign of the sum in example c above?
   What is the sign of the sum in example d above?

   What is the relationship of the absolute value of the sum to the absolute values of the addends in each of these examples?

5. Elicit that the rules for addition in the set of integers seem to apply for addition in the set of rationals. Tell pupils that this is true.

6. Using the rules developed above, find the following sums. Check the answers on the number line.

   $4\frac{1}{4} + 2\frac{1}{4}$

   $6 + (-3\frac{1}{2})$

   $-2 + (-2\frac{1}{2})$

   $-7\frac{1}{3} + 4\frac{2}{3}$

7. Return to challenge. Using the rules for addition of rational numbers, find the sum of $\frac{5}{8} + (-\frac{1}{4})$. Solution: $\frac{5}{8} + (-\frac{2}{8}) = \frac{3}{8}$.

II. Practice

A. Find the following sums:

   $3\frac{1}{2} + (-2)$

   $\frac{3}{5} + 0$

   $\frac{1}{4} + (-\frac{3}{4})$

   $\frac{7}{8} + \frac{3}{4}$

   $7\frac{1}{2} + (-7\frac{1}{2})$

   $\frac{3}{10} + (-\frac{9}{10})$

B. Show by illustrations that the set of rational numbers has the same properties of addition as the set of integers

III. Summary

A. Explain how the sum of two integers is determined when the addends are both positive; when the addends are both negative; when one addend is positive and one addend is negative.

B. How do the rules for adding rational numbers compare with the rules for adding integers?

Lesson 58

Topic: Subtraction of Positive and Negative Rational Numbers

Aim: To develop rules for subtraction in the set of rational numbers

Specific Objectives:

To review the rules for subtraction in the set of integers
To apply these rules to subtraction in the set of rational numbers

Challenge: Which answer will be greater? $6\frac{3}{4} + (-2\frac{1}{4}) = ?$ or $6\frac{3}{4} - (-2\frac{1}{4}) = ?$

I. Procedure

A. Review rules for subtraction in the set of integers

1. Have pupils recall that by definition, in the set of integers, subtracting a number is the same as adding its opposite.

2. Rewrite these subtraction exercises as related addition examples. Find their sums.

$8 - 12 = 8 + (?)$ $\quad$ $-4 - (-8) = -4 + (?)$

$-5 - 3 = -5 + (?)$ $\quad$ $9 - (-3) = 9 + (?)$

B. To apply the rules for subtraction to the set of rational numbers

1. Elicit that we defined the subtraction of an integer as the addition of its opposite.

2. Tell pupils that just as the rules for addition of integers were applied to addition of rational numbers, so we can apply the rule for subtraction of integers to subtraction in the set of rational numbers.

3. Consider these three different examples:

$\frac{6}{7} - \frac{2}{7} = ?$ $\quad$ $-\frac{6}{7} - \frac{2}{7} = ?$ $\quad$ $\frac{6}{7} - (-\frac{2}{7}) = ?$

a. Elicit that these are examples of subtraction of rational numbers.

b. Write the opposites of $\frac{2}{7}$, of $-\frac{2}{7}$.

c. Rewrite the subtraction examples as addition examples and find the answers.

4. Have pupils realize that the operation of subtracting a rational number is defined as adding its opposite.

5. Answer the challenge.

II. Practice

A. Perform the following operations:

$5 - (-8) = ?$ $-68 - (-78) = ?$

$24 - 4 = ?$ $-9 - 8 = ?$

B. Compute the answers to:

$\frac{5}{3} - \frac{3}{5}$ $-\frac{3}{10} - \frac{1}{3}$

$6 - (-\frac{5}{8})$ $-\frac{6}{11} - (-\frac{6}{11})$

C. Compute: $\frac{5}{8} - \frac{3}{16}$.

To what set of numbers does the answer belong?

D. Is $\frac{5}{8} - \frac{3}{8} = \frac{3}{8} - \frac{5}{8}$ true?

What does this show about the commutative property with respect to the operation of subtraction?

How many counter examples are needed to show that a property does not hold? (one)

E. Is the set of rational numbers closed with respect to subtraction?

III. Summary

A. What is the rule for the subtraction of rational numbers?

B. Show by example that the commutative and associative properties do not hold for subtraction in the set of rational numbers.

Lesson 59

Topic: Multiplication of Rational Numbers

Aim: To develop rules for multiplication of rational numbers

Specific Objectives:

To review the rules for multiplication in the set of integers
To apply these rules to multiplication in the set of rational numbers

Challenge: Name two rational numbers whose product is -42.

I. Procedure

A. Review rules for multiplication in the set of integers

1. Have pupils recall the rules for the multiplication of integers.

2. Find the products of:

a. $7 \times (-6)$

b. -9×3

c. $-4 \times (-5)$

d. 8×10

B. Multiplication in the set of rational numbers

1. Consider the example above: $7 \times (-6) = -42$.

a. Is 7 a rational number? Explain.

A rational number is any number which can be expressed as an integer divided by any integer except zero.

b. Is -6 a rational number? Explain. $(-\frac{6}{1}, -\frac{12}{2}, \ldots)$

c. Refer to challenge: $7 \times (-6) = -42$ is one answer to the challenge since 7 and -6 are rational numbers.

2. Using similar procedures for A-2b, c, d above, guide pupils to see that the rule of signs in multiplication in the set of integers seems to apply to the set of rational numbers. Tell pupils that this is true.

3. Find the products:

$\frac{1}{5} \times 5 = ?$ $\quad -6 \times (-\frac{2}{3}) = ?$ $\quad 10 \times (-\frac{1}{2}) = ?$ $\quad -\frac{3}{4} \times \frac{1}{8} = ?$

4. In each example above,

a. How was the sign of the product determined?
b. How was the absolute value of the product obtained?

II. Practice

A. Find the following products:

$8 \times (-3)$

-3×6

$-12 \times (-15)$

17×3

$-14 \times (-2)$

$-16 \times (-20)$

B. Find the following products:

$-3 \times \frac{1}{2}$

$\frac{11}{13} \times (-1)$

$7 \times (-\frac{3}{2})$

$-\frac{3}{4} \times (-\frac{5}{6})$

$\frac{1}{3} \times 0$

$\frac{2}{5} \times 1$

C. Find the following products:

$3\frac{1}{2} \times (-\frac{1}{4})$

$-4\frac{2}{3} \times 1\frac{1}{2}$

$-5\frac{1}{2} \times (-1)$

$-3.5 \times (-.5)$

$-2.5 \times (-.02)$

$-.03 \times (-.65)$

D. Show by illustrations that under the operation of multiplication, the set of rational numbers has the same properties as the set of integers.

III. Summary

A. Why is an integer considered to be a rational number?

B. Compare the rules for multiplication in the set of rational numbers with the rules for multiplication in the set of integers.

Lesson 60

Topic: Division of Rational Numbers

Aim: To develop rules for division in the set of rational numbers

Specific Objectives:

To review division in the set of integers
To apply these rules to the set of rational numbers

Challenge: Write $\frac{3}{4} \div \frac{5}{7}$ as a related multiplication example.

I. Procedure

A. Review rules for division of integers

1. Have pupils recall the rule of signs in division of integers.

2. Have pupils do the following examples:

$(+8) \div (+5) = ?$ $\qquad$ $(+6) \div (-2) = ?$

$(-3) \div (+1) = ?$ $\qquad$ $(-15) \div (-3) = ?$

B. Rules of division in the set of rational numbers

1. Refer to challenge. How do we determine the quotient of $\frac{3}{4}$ divided by $\frac{3}{5}$?

Recall that since the positive rationals behave like the fractional numbers of arithmetic, the division can be performed as follows:

$\frac{3}{4} \div \frac{7}{5} = \frac{3}{4} \times \frac{5}{7}$ Why? (multiplicative inverse)

$= \frac{3 \times 5}{4 \times 7}$ Why? (definition of multiplication of rational numbers)

$= \frac{15}{28}$

2. Guide pupils to see that just as the rules for addition, multiplication, and subtraction of integers were applied to the set of rational numbers, so we can apply the rule for division of integers to the set of rational numbers.

3. Have pupils apply this rule to find the following quotient: $\frac{3}{4} \div (-\frac{7}{5})$.

What is the sign of the quotient? Why?

$\frac{3}{4} \div (-\frac{7}{5}) = \frac{3}{4} \times (-\frac{5}{7})$ Why?

$= -\frac{3 \times 5}{4 \times 7}$ Why?

4. Check by multiplication. Is $-\frac{15}{28} \times -\frac{7}{5} = \frac{3}{4}$ true?

5. Compute:

$\frac{3}{4} \div (-\frac{3}{5})$

$\frac{3}{4} \div (-\frac{3}{5}) = \frac{3}{4} \times (-\frac{5}{3})$ Why?

$= -\frac{3 \times 5}{4 \times 3}$ Why?

$= -\frac{3 \times 5}{3 \times 4}$ Why?

$= -\frac{5}{4}$ Why?

6. After several similar examples, elicit that the rules for signs in division in the set of rational numbers is the same as for division of integers.

II. Practice

A. Find the following quotients:

1. $\frac{3}{11} \div \frac{5}{22}$
2. $8 \div (-\frac{1}{4})$
3. $-\frac{3}{10} \div \frac{5}{8}$
4. $-\frac{2}{5} \div (-\frac{2}{5})$

B. Check each of the examples in A, by multiplication.

III. Summary

A. What is the reciprocal of a rational number?

B. What is the rule for determining the sign of the answer in division of rational numbers?

CHAPTER VII

EQUATIONS AND INEQUALITIES

In this chapter are materials and suggested procedures for:

- solving open sentences in one variable - equations and inequalities
- using equations to solve verbal problems
- reinforcing the meaning of ratio and equivalent ratios
- developing the meaning of proportion
- using proportion in problem solving

The chapter presents a means of reviewing the concepts of open sentence, variable, replacement set, solution set. Then, the basic concept of equivalent equations is introduced. Pupils are guided to learn that two equations in the same variable are equivalent if they have the same replacement set and the same solution set. If, for example, the replacement set is the set of rational numbers, $3x+6=18$ and $x+2=6$ are equivalent equations, because the solution set for each is $\{4\}$.

In solving an equation, the aim is to use systematic procedures to transform an open sentence such as $2x+7=17$ into an equivalent equation of the form $x=5$. This latter sentence has the distinct advantage of having a solution set which is obvious. Procedures are suggested by which pupils can discover and use the two basic properties of equality, the addition property and the multiplication property for transforming an equation into an equivalent equation.

In this chapter also are suggestions for developing understanding and skill in solving various problem situations through the use of simple equations. The distinction between the solution of an equation and the solution of a problem is stressed. The solution of an equation is a number which must be interpreted by the pupil in terms of the problem situation.

The concept of a ratio as an ordered pair and the concept of equivalent ratios are reviewed. Proportion is then introduced as a mathematical sentence which says two ratios are equivalent. A proportion may be presented as an open sentence or as a statement which is either true or false.

Pupils are led to discover that in a proportion "the product of the means equals the product of the extremes." Using this fact as a tool, they learn to find the solution of various verbal problems through the use of proportion.

CHAPTER VII

EQUATIONS AND INEQUALITIES

Lessons 61-78

Lesson 61

Topic: Open Sentences

Aim: To reinforce the concepts needed for finding solutions of open sentences.

Specific Objectives:

To review and reinforce the meaning of:
- Statements of equality
- Open sentences
- Variable
- Replacement set
- Solution set

Challenge: Which of these is an open sentence?

a. $4 + 2 = 6$ b. $10 - 3 = 7$ c. $2n = 14$

I. Procedure

A. Statements of equality

1. Refer to challenge. Which of the above sentences can be judged true or false? (a and b) How do mathematicians refer to a sentence which can be judged true or false? (It is called a statement.) Which of the above sentences are statements? (a and b)

2. Why isn't $2n = 14$, the sentence in c, considered a statement? (We cannot judge its truth or falsity until n is replaced by the name of the number.)

B. Open sentences

1. What name do we give to a sentence such as $2n = 14$ whose truth or falsity cannot be determined until n is replaced by the name of a number? (open sentence)

2. Which of the following are open sentences and which are statements?

a. $5 \times 3 = 15$
b. $n = 2$
c. $4 + 8 = 12$
d. $3 + \square = 13$
e. $y + 3 = 10$
f. x is an odd number

C. Variables

1. Name the symbol in each of the open sentences in B-2 which holds the place for a member of the replacement set. What are such symbols called? (variables)

2. What is the variable in each of the following open sentences?

   a. $\square + 4 = 8$
   b. She has blue eyes.
   c. $x + 3 = 14$
   d. $5 + y = 2$

D. Replacement set

1. In the open sentence $3 + x = 10$, let us agree that x refers to any one of the numbers 0,1,2,3,4,5,6,7,8,9,10. Then we call $\{0,1,2,\ldots,10\}$ the replacement set. Which elements of the replacement set will make the sentence true? (7)

2. If the replacement set for each of the following sentences is $\{0,1,2,3,4,5,\ldots\}$, what are all replacements that will make each of these sentences a true statement?

   a. $n + 2 = 3$ (1)
   b. $3y = 9$ (3)

3. Tell pupils that in the open sentence $n + 2 = 3$, since 1 is the member of the replacement set that will make the sentence true, we say that 1 is a root, or that 1 satisfies the equation. Therefore, 1 is a solution of the equation $n + 2 = 3$.

   Which member of the replacement set is a root of the equation $3y = 9$? What is the solution of the equation $3y = 9$?

E. Solution set

1. We say we have found the solution set of an open sentence, when, for a given replacement set, we have found all the replacements for the variable that result in a true statement.

   Refer to examples D-2, a,b.

   a. What do we call the number that makes the sentence $n + 2 = 3$ true? (a solution of the sentence) Is there any other replacement which will make the sentence true? (no) What then is the solution set of $n + 2 = 3$? Answer: $\{1\}$.

   b. Is 3 a root of the open sentence $3y = 9$? What other replacements will make the sentence true? (none) How do we indicate that 3 is the only replacement which will make the sentence true? Answer: $\{3\}$.

   c. What name do we give to $\{3\}$? (solution set)

d. How is the solution set related to the replacement set? (It is a subset of the replacement set.)

2. Use $\{0,1,2,3,\ldots\}$ as the replacement set for the variable to find the solution set for each of these open sentences.

a. $n + 5 = 12$

b. $\frac{n}{3} = 3$

c. $y = 4$

d. $2x + 1 = 11$

II. Practice

A. Which of the following are open sentences and which are statements? Explain.

1. $7 + 5 = 12$

2. $n + 5 = 12$

3. $4 + (-1) = 3$

4. $10 + x = 15$

5. $4 - \square = 2$

6. $8 \div 4 = 2$

B. Circle the variable in each of the following open sentences.

1. She is wearing a red dress.

2. $\frac{1}{2}$ of $\square = 5$

3. $x + 5 = 8$

4. $y = 10$

5. $4a + 2 = 10$

6. It is made of copper.

C. For each of the following sentences consider $\{-1,0,1,2,3\}$ as the replacement set for the variable. Find the solution set of each sentence.

1. $3x = -3$ $\{-1\}$

2. $4 + y = 7$ $\{3\}$

3. $y \times 0 = 0$ $\{-1,0,1,2,3\}$

4. $b = -1$ $\{-1\}$

5. $3x = 15$ $\{\ \}$ or $\emptyset$

III. Summary

A. When is a sentence a statement?

B. What is meant by an open sentence? Give an example of one.

C. A variable is sometimes referred to as a "placeholder." Can you explain why? (It holds the place in a sentence for a name of some member of a set.)

D. How is the solution set of an open sentence related to its replacement set? (It is a subset of the replacement set.)

Lesson 62

Topic: Inequalities

Aim: To review the solution of an inequality by testing the members of a given replacement set

Specific Objectives:

- To review the meaning of symbols of inequality
- To judge whether an inequality is true or false
- To reinforce the concept that an open sentence may be an inequality
- To review the solution of inequalities of the form $x > n$ and of $x < n$

Challenge: How many elements belong to the solution set of $x > 4$, when the replacement set is $\{1,3,5,7,9\}$?

I. Procedure

A. Review meaning of symbols of inequality

1. What does $5 > 3$ mean? $x < 4$? $6 \neq 5$?

Note: Have pupils realize that the symbols ">" and "<" always point to the smaller value.

2. Recall that a sentence which says that one quantity does not equal another quantity is called an inequality.

B. Judging truth and falsity

1. Which of the following are true statements? (a,b,f)

a. $8 \neq 5$

b. $-7 < 8$

c. $9 > 14$

d. $-6 > 0$

e. $5 \neq 3 + 2$

f. $58 > -57$

2. Using a symbol of equality or inequality, change each false statement in 1 above to a true statement. (c. $9 < 14$, d. $-6 < 0$, e. $5 = 3 + 2$)

C. An open sentence may be an inequality

1. Consider the following sentences.

a. $2x = -8$

b. $2 > -1$

c. $w + 1 < 3$

d. $3 \neq 1 + 2$

e. $\Delta + 8 = 17$

f. $5x > 0$

g. $15 > 14 + y$

h. $3a + 7 \neq 15$

i. $-1 + 4 < 10$

Which are inequalities? Which are equalities?

2. Which are open sentences? Elicit that an open sentence may be an inequality, as c,f,g,h, above.

D. Review the solution of inequalities by testing the members of the replacement set.

1. Recall that all the members of the replacement set which make the sentence true form the solution set of the open sentence.

2. Use the replacement set $\{-4,-3,-2,-1,0,1,2,3\}$ and find the solution set of each of the following:

a. $x > 2$ $\{3\}$

b. $x > -1$ $\{0,1,2,3\}$

c. $x < 3$ $\{-4, -3, \ldots, 2\}$

d. $x < 0$ $\{-4, -3, -2, -1\}$

e. $x < -4$ $\{\ \}$ or $\emptyset$

3. Referring to the challenge, find the solution set of the inequality $x > 4$ when the replacement set is $\{1,3,5,7,9\}$. Answer: the solution set is $\{5,7,9\}$. Therefore, the solution set contains 3 elements.

II. Practice

A. Find the solution set of each of the following inequalities from the replacement set $\{0,1,2,3,4,5\}$.

1. $x < 5$
2. $x > 1$
3. $x < -4$
4. $x > 3$
5. $x < 0$
6. $2x < 6$
7. $3x > 12$
8. $2x > 6$

B. If the replacement set for the examples in A is $\{-5,-4,-3,-2,-1,0,1,2,3,4,5\}$, find the solution set of each inequality.

C. If the replacement set for the examples in A is the set of integers, find the solution set of each inequality.

III. Summary

A. What are three symbols of inequality? Explain the meaning of each.
B. In an inequality, does the symbol ">" point toward the larger or the smaller value?
C. What kind of a sentence is the inequality $x + 1 > 5$? (open)
D. Show by an example that a change in the replacement set may cause a change in the solution set of an inequality.

Lesson 63

Topic: Inequalities

Aim: To find the solution set of an inequality of the form $ax + b > c$, or $ax + b < c$

Specific Objectives:

To recall that the solution set of an open sentence may be a finite set, an infinite set, or the empty set
To solve inequalities of the form $ax + b > c$ or $ax + b < c$

Challenge: Using the set of positive even integers as a replacement set, find the solution set for the inequality $2x + 4 < 14$

I. Procedure

A. The solution set of an inequality may be finite, infinite, or the empty set

1. Using as the replacement set for x the set of positive odd integers, find the solution set of $2x < 10$ $D = \{1,3,5,...\}$

   Note to Teachers: Pupils should realize that there is no point in testing more elements of the replacement set after 5, since twice each succeeding element is greater than 10.

   a. What is the solution set for $2x < 10$? Answer: $\{1,3\}$

   b. How many elements are there in the solution set? (2)

   c. Is the solution set finite, infinite, or empty? (finite)

2. Find the solution set of $3x > 3$ $D = \{1,3,5,...\}$

   a. What is the solution set for $3x > 3$? Answer: $\{3,5,7,...\}$

   b. How many elements are there in the solution set?

   c. How may we classify the solution set? (an infinite set)

3. Find the solution set of $5x < 0$ $D = \{1,3,5,...\}$

   a. By trial, are there any elements of the replacement set which make the open sentence true? (no)

   b. How many elements are there in the solution set? (none)

c. How may we classify the solution set? (The solution set is the null set or empty set and may be written in symbols as $\{\ \}$ or $\emptyset$.)

d. If the replacement for x were -1, we would have a true statement, since $(5)(-1) = -5$ and $-5 < 0$. Why is -1 not an element of the solution set? (Because -1 is not an element of the replacement set, and the solution set is a subset of the replacement set.)

B. Solving inequalities of the form $ax + b > c$, $ax + b < c$

1. Return to the challenge. What is the replacement set? Answer: $\{2,4,6,\ldots\}$

2. If $x = 2$, what is the value of $2x + 4$? ($2 \times 2 + 4$ or $4 + 4$ or 8) Is this less than 14? (Yes) Then, $2x + 4 < 14$ is true when $x = 2$. Therefore, 2 is an element of the solution set.

3. If $x = 4$, what is the value of $2x + 4$? ($2 \times 4 + 4$ or $8 + 4$ or 12) Does the replacement of 4 for x make the inequality $2x + 4 < 14$ true? (Yes, $12 < 14$) Therefore, 4 is also an element of the solution set.

4. If $x = 6$, what does $2x + 4$ equal? ($2 \times 6 + 4$ or $12 + 4$ or 16) Is $2x + 4 < 14$ true when $x = 6$? (No) Therefore, 6 is not a member of the solution set.

5. Why is it not necessary to try any more elements in the replacement set? (As we try numbers greater than 6, the value of the expression $2x + 4$ increases beyond 14.) Answer the challenge.

II. Practice

A. Use the replacement set $\{-3, -2, -1, 0, 1, 2, 3\}$ to find the solution set of

$2x + 1 > 2$ $\{1, 2, 3\}$

$3x + 2 < 7$ $\{-3, -2, -1, 0, 1\}$

$5x - 2 > 14$ $\{\ \}$ or $\emptyset$

B. Use as the replacement set the set of non-negative integers to find the solution set for each inequality in A.

C. If the replacement set is $\{1, 2, 3, \ldots\}$, write the solution set for

$3x + 1 < 3$ $\{\ \}$ or $\emptyset$

$2x + 3 > 10$ $\{4, 5, 6, \ldots\}$

$3x - 2 < 9$ $\{1, 2, 3\}$

D. Use as the replacement set the set of integers to find the solution set of each inequality in C.

III. Summary

A. What is the maximum number of elements that could be in a solution set? (an infinite number)

B. What is the least number? (none)

C. Can the solution set have elements that are not in the replacement set? (no) Explain. (The solution set is a subset of the replacement set.)

Lesson 64

Topic: Equations

Aim: To develop the concept of equivalent equations

Specific Objectives:

To understand the meaning of equivalent equations
To identify equivalent equations
To form sets of equivalent equations

Challenge: Use $\{1,2,3,4,5\}$ as the replacement set to find the solution set for each of the following equations:

$$3x + 5 = 14$$
$$3x = 9$$
$$x + 1 = 4$$

What have the solution sets in common?

I. Procedure

A. Meaning of equivalent equations

1. Refer to challenge. Have pupils observe that for each of the equations presented in the challenge the solution set is $\{3\}$.

2. Using the same replacement set, have pupils find the solution set for other similar sets of equations, e.g.,

a. $2y = 8$
$y+5 = 9$
$y = 4$

b. $3w - 1 = 5$
$w + 8 = 10$
$2w = 4$

3. Elicit that for each set of equations the solution set is the same.

4. Tell pupils that if two (or more) equations of the type presented above, in the same variable, and with the same replacement set, have the same solution set, they are called equivalent equations.

B. Identifying equivalent equations

1. Consider the following equations. Which, if any, are equivalent? Use the set of positive integers as the replacement set.

a. $x + 2 = 8$ b. $x - 2 = 3$ c. $3x = 18$

(The solution set of equation a is $\{6\}$; of equation b is $\{5\}$; of equation c is $\{6\}$.)

Since equations a and c have the same solution set, they are equivalent. Equation b is not equivalent to either of the other two.

2. In each of the following exercises, identify the two equations that are equivalent. Use $\{0,1,2,3,4,5\}$ as the replacement set.

a. $x + 17 = 19$
$x - 3 = -1$
$x + 8 = 9$

b. $6z = 12$
$2z = 6$
$3z = 9$

C. Forming sets of equivalent equations

1. Using the set of non-negative integers as a replacement set, find the solution set for $2x = 4$.

   Have pupils suggest equivalent equations. ($x = 2$, $4x = 8$, $x + 2 = 4$, etc.)

   Record suggestions on the chalkboard and have class test for equivalence by using the replacement set.

2. Using the same replacement set, find the solution set for $w + 6 = 10$. Have pupils suggest equivalent equations, record the suggestions on the chalkboard and test.

3. Have pupils note that a given equation may have many equivalent equations.

II. Practice

A. Match the equation in column a with its equivalent equation in column b. Use $\{-3, -2, -1, 0, 1, 2, 3\}$ as the replacement set.

a	b
$n + 2 = 1$	$n + 2 = 2$
$3n = 0$	$2n = 6$
$n - 1 = 2$	$2n = -2$

B. Using the replacement set above, select the equation that is not equivalent to the other three.

$4x + 1 = 13$ $\quad$ $2x = 6$ $\quad$ $x + 4 = 9$ $\quad$ $x = 3$

C. Using the set of non-negative integers as a replacement set, write for each of the following equations, 3 equivalent equations.

$5x = 20$ $\quad$ $x + 8 = 17$ $\quad$ $2x + 1 = 11$ $\quad$ $3x - 2 = 13$

III. Summary

A. What is meant by equivalent equations?

B. What method do we use to tell whether two equations are equivalent?

C. How many equations can you find which are equivalent to a given equation?

Lesson 65

Topic: Equations

Aim: To form equivalent equations by the use of the addition property of equality

Specific Objectives:

To understand that an equivalent equation may be formed by the addition of the same quantity to both members of an equation
To form equivalent equations by adding the same number to both members of an equation
To recognize an equation whose solution set is obvious

Challenge: Using the set of integers as the replacement set, find the solution set for each of the following equations.

$x + 3 = 6 \qquad x = 3 \qquad x + (-1) = 2$

How are these equations related?
In which of these is the solution set easiest to find?

I. Procedure

A. Equivalent equations formed by addition

1. Consider the following equations:

$n + 3 = 9$
$n + 4 = 10$
$n + 5 = 11$
$n + 6 = 12$

a. If the replacement set is $\{1,2,3,\ldots 10\}$, what is the solution set of each equation? Are they equivalent equations? Why?

b. How does the left member of equation $n + 4 = 10$ compare with the left member of $n + 3 = 9$? $(n + 4 = n + 3 + 1)$ How do their right members compare? $(10 = 9 + 1)$

c. How does the left member of equation $n + 3 = 9$ compare with the left member of $n + 4 = 10$? $(n + 3 = n + 4 + (-1))$ How do their right members compare? $(9 = 10 + (-1))$

d. Through similar questions for the remaining equations, elicit that when the same quantity (positive or negative) is added to both members of an equation an equivalent equation is formed.

e. After working in the same way with several other sets of equivalent equations, lead pupils to generalize that if $a = b$ then $a + c = b + c$. Tell pupils that this is referred to as the addition property of equality.

B. To form equivalent equations by the use of the addition property of equality

1. Consider the equation $x + 2 = 5$. Replacement set is the set of integers.

   Using the generalization arrived at in A, have pupils suggest equivalent equations formed by the addition of a positive or negative integer to both members of the equation.

2. Record suggestions on the chalkboard.

3. Have pupils test by replacement whether the suggested equations are equivalent to $x + 2 = 5$.

C. To recognize an equation whose solution set is obvious

1. Refer to challenge. Elicit that the equations are equivalent.

2. The solution set of equation b is easiest to find because it is obvious that the only replacement for x that will result in a true statement is 3. The solution set of an equation is obvious when the variable appears alone as one member.

II. Practice

A. What number was added to each member of an equation in column a to form the equivalent equation in column b?

a	b
$x + 4 = 9$	$x + 6 = 11$
$x - 1 = 4$	$x + 1 = 6$
$x + 3 = 8$	$x + 2 = 7$

B. Write an equivalent equation to $x + 5 = 10$ by adding:

4 to each member

-4 to each member

C. Which of the following equations have solution sets which are obvious?

$3x + 5 = 10$

$w = 2$

$y - 8 = 15$

$n + 3 = -25$

$a = 72$

$2b = 16$

III. Summary

A. When the same number is added to both members of an equation to form a new equation, how are the equations related?

B. What kind of a number can be added to both members of an equation to obtain an equivalent equation?

C. When is the solution set of an equation obvious?

D. What is meant by the addition property of equality?

Lessons 66 and 67

Topic: Equations

Aim: To use the addition property of equality to solve equations

Specific Objectives:

To review the meaning of the additive identity: $a + 0 = a$
To review the meaning of the additive inverse: $a + (-a) = 0$
To solve and check equations of the form $x + a = b$ $(a > 0)$ using the addition property of equality

Challenge: Transform $y + 2 = 11$ into an equivalent equation whose solution is obvious.

I. Procedure

A. Review the meaning of the additive identity

1. Complete each of the following:

$4 + 0 = ?$
$0 + 3 = ?$
$29 + 0 = ?$
$x + 0 = ?$

2. Which number, when added to any given number, results in a sum which is the given number? (zero)

Tell the pupils that zero is called the identity element of addition.

B. Review additive inverse

1. Complete the following:

$3 + (-3) = ?$
$-5 + 5 = ?$
$23 + (-23) = ?$

What is the sum in each case? (zero)
How is the first number of each pair related to the second? (They are opposites.)

2. Elicit that if the sum of two numbers is zero, each is called the additive inverse of the other.

3. Replace the frame to make each of the following a true statement.

$2 + \square = 0$ $\qquad$ $\square + 18 = 0$

$-7 + \square = 0$ $\qquad$ $\square + 0 = 0$

Elicit that starting with a given number, we can obtain a sum of zero when the additive inverse is added to the number.

Note: The additive inverse of zero is zero.

C. Solving equations of the form $x + a = b$ $(a > 0)$ by the addition property

Note: Unless otherwise specified, the replacement set for the variable in the equations in this chapter is the set of integers.

1. What is the replacement for $\square$ in each of the following that will result in a true statement?

a. $7 + (3 + \square) = 7$ $\qquad$ d. $(\square + 2) + 20 = 20$

b. $9 + (5 + \square) = 9$ $\qquad$ e. $x + (3 + \square) = x$

c. $(-1 + \square) + 8 = 8$ $\qquad$ f. $(-5 + \square) + y = y$

Note: Pupils should be able to justify their choice of a replacement for the frame. For example, in $7 + 3 + \square = 7$, the replacement for $\square$ is -3 because:

$$\begin{aligned} 7 + (3 + -3\ ) &= 7 \\ 7 + \quad 0 \quad &= 7 \\ 7 &= 7 \end{aligned}$$

Elicit that in each case, the additive inverse is used.

2. Consider two equations:

$$\begin{aligned} x + 5 &= 12 \\ x &= 7 \end{aligned}$$

a. How are the equations related? (They are equivalent.) In which is the solution obvious? $(x = 7)$

b. Using what we have just learned, how can we transform the given equation into an equivalent equation whose solution is obvious?

$$\begin{array}{rcll} x + 5 &=& 12 & \\ x + 5 + (-5) &=& 12 + (-5) & \text{Addition property} \\ x + 0 &=& 7 & \text{Additive inverse} \\ x &=& 7 & \text{Additive identity} \end{array}$$

c. What is the solution of $x = 7$? (7)
Since $x = 7$ and $x + 5 = 12$ are equivalent, what is the solution of $x + 5 = 12$? (7)

d. Checking the solution:

1) $x + 5 = 12$
$7 + 5 = 12$ True statement

Therefore, 7 is the solution of $x + 5 = 12$.

2) Have pupils recall that in an equation such as $x + 5 = 12$, we say we have solved the equation when the replacement for the variable makes the statement true.

3) Now have pupils answer the challenge.

3. Consider the equation $16 = 13 + x$.

a. How can we arrive at the equivalent equation where x alone is the right member? (add -13 to both members of the equation)

Why did we choose -13 to add to both members of the equation? (If we add -13 and 13, the result will be the additive identity, (0), and x will then stand alone as the right member.

Have pupils give reasons for each step.

$$\begin{array}{rcl} 16 &=& 13 + x \\ 16 + (-13) &=& 13 + (-13) + x \\ 3 &=& 0 + x \\ 3 &=& x \end{array}$$

b. Since $3 = x$ and $16 = 13 + x$ are equivalent, what is the solution of $16 = 13 + x$? (3)

c. How would you check the solution of $16= 13 + x$?

II. Practice

A. State what number must be added to each member of the equation. Find the solution and check your answer.

1. $x + 5 = 8$
2. $y + 3 = 12$
3. $a + 9 = 7$
4. $2 + r = 13$
5. $0 + x = 7$
6. $15 = n + 7$
7. $x + (-4) = 6$
8. $5 + x = 0$
9. $y + 9 = 13$
10. $y + (-2) = 8$

III. Summary

A. What is the additive identity in the set of integers?

B. What is meant by the additive inverse of a number?

C. What is the additive inverse of zero?

D. If the replacement set is large, what disadvantage is there in using the replacement method to solve an equation?

E. What better method exists for solving an equation? (finding an equivalent equation whose solution is obvious)

F. In equations such as those we have studied today, how can we tell what must be added to both members of the equation so that an equivalent equation results whose solution is obvious?

G. How can we be sure that the solution set of the "obvious" equation is also the solution set of the given equation?

Lesson 68

Topic: Equations

Aim: To use the addition property to solve equations

Specific Objectives:

To review subtraction involving addition of additive inverse
To solve and check equations of the form $x - a = b$ $(a > 0)$

Challenge: Write an equivalent equation to $10 = x - 4$ by expressing the right member in terms of addition.

I. Procedure

A. Review subtracting as the adding of the additive inverse

1. Which number is the additive inverse of each of the following:

10 -2 24 x $-y$

2. Rewrite each of these subtraction sentences as an addition sentence.

$4 - 2 = 2$ $(4 + (-2) = 2)$

$9 - 14 = -5$ $(9 + (-14) = -5)$

$5 - (-7) = 12$ $(5 + (+7) = 2)$

3. Elicit that subtracting a number is the same as adding its opposite or additive inverse.

B. Solving and checking equations of the form $x - a = b$ $(a > 0)$

1. How would you express each of the following in terms of addition?

$x - 3$ $y - 4$ $a - 10$ $b - 15$

2. What is the replacement for $\square$ in each of the following that will result in a true statement?

$7 + (-5) + \square = 7$ $x + (-3) + \square = x$

$9 + (-2) + \square = 9$ $y + (-4) + \square = y$

3. Consider the equation: $x - 3 = 8$.

a. Write an equivalent equation expressing the left member in terms of addition. $(x + (-3) = 8)$

b. Have pupils use the addition property to solve the equation as follows.

$$\begin{aligned} x - 3 &= 8 & & \\ x + (-3) &= 8 & & \text{Why?} \\ x + (-3) + 3 &= 8 + 3 & & \text{Why?} \\ x + 0 &= 11 & & \text{Why?} \\ x &= 11 & & \text{Why?} \end{aligned}$$

What is the solution of $x = 11$? of $x - 3 = 8$?

c. How would you check the solution of $x - 3 = 8$?

4. Have pupils answer the challenge, solve the equation, and check.

II. Practice

A. Express each of the following subtractions as an addition:

1. $y - 4$ $(y + (-4))$
2. $x - 19$
3. $a - 3$
4. $r - 14$
5. $n - (-11)$ $(n + (11))$
6. $b - (-25)$

B. Which number would you add to both members of each of the following equations to solve it? How did you decide?

1. $y - 15 = 10$
2. $x - 4 = 8$
3. $x - 5 = -1$
4. $8 = r - 9$
5. $b - 4 = 2$
6. $-10 = x - 8$
7. $y - 1 = 18$
8. $y - 5 = 0$

C. Solve and check each of the equations in B.

III. Summary

A. Give an illustration to show that subtracting a number is equivalent to adding its opposite.

B. What would be the first step in solving an equation such as: $x - 7 = 10$? (Express the subtraction as addition of the additive inverse.)

C. What principle is then used in completing the solution?

Lesson 69

Topic: Equations

Aim: To form equivalent equations by multiplication

Specific Objectives:

To understand that an equivalent equation may be formed by multiplying both members of the equation by the same quantity

To form equivalent equations by multiplying both members of the equation by the same quantity using the multiplication property of equality, if $a = b$, then $a \times c = b \times c$

Challenge: Using the set of rational numbers as the replacement set, are the following equations equivalent?

$$\begin{aligned} 2n &= 4 \\ n &= 2 \\ \tfrac{1}{2}n &= 1 \\ -4n &= -8 \end{aligned}$$

I. Procedure

A. Equivalent equations formed by multiplication

1. Refer to challenge. Why are the equations equivalent? (because the solution set for n in each equation is 2)

2. How does the left member of $2n = 4$ compare with the left member of $n = 2$? ($2n = 2 \times n$) How do their right members compare? ($4 = 2 \times 2$)

   Note: It is customary to use parentheses in cases where the symbol for multiplication (x) might be confused with the symbol for the variable x.

3. How does the left member of equation $n = 2$ compare with the left member of $2n = 4$? $n = \frac{1}{2}(2n)$ How do their right members compare? $2 = \frac{1}{2}(4)$

4. By what number must both members of $\frac{1}{2}n = 1$ be multiplied to obtain $n = 2$? (2)

5. How does the left member of $-4n = -8$ compare with the left member of $2n = 4$? ($-4n = -2 \times 2n$) How do their right members compare? ($-8 = -2 \times 4$)

6. Elicit that when both members of an equation are multiplied by the same non-zero number an equivalent equation is formed.

B. To form equivalent equations by multiplication

1. Consider the equation $12a = 36$. Have pupils suggest equivalent equations using the multiplication property of equality. For example: $2(12a) = 2(36)$; $\frac{1}{2}(12a) = \frac{1}{2}(36)$.

2. Record suggestions on the chalkboard.

3. Have pupils test by replacement whether the suggested equations are equivalent to $12a = 36$.

II. Practice

A. By what number is each member of an equation in column A multiplied to form the equivalent equation in column B?

A	B
$5x = 10$	$20x = 40$
$3w = 9$	$-6w = -18$
$-2r = -6$	$6r = 18$
$\frac{1}{3}y = 3$	$y = 9$
$4c = 9$	$2c = 4.5$
$-7b = 14$	$b = -2$

B. Write an equivalent equation to $15d = 60$ by multiplying each member by 4, by -2, by $\frac{1}{3}$, by $-\frac{1}{5}$.

III. Summary

A. When both members of an equation are multiplied by the same number to form a new equation, how are the equations related?

B. By what kind of a number can both members of an equation be multiplied to obtain an equivalent equation? (Any number, except zero)

C. The multiplication property of equality states that if $a = b$, then $a \times c = b \times c$ (for all a, b, c). Give an illustration of this property.

Lesson 70

Topic: Equations

Aim: To use the multiplication property of equality to solve equations

Specific Objectives:

To review the multiplicative identity
To review multiplicative inverses
To solve and check equations of the type $ax = b$ (a, b are rational numbers and $a \neq 0$) using the multiplication property of equality

Challenge: By what numeral should n be replaced if each of the following is to be a true statement:

$5 \times n = 5$ $\qquad$ $\frac{1}{2} \times n = \frac{1}{2}$

$-10 \times n = -10$ $\qquad$ $.05 \times n = .05$

I. Procedure

A. Review the identity element of multiplication

1. Elicit that each of the open sentences in the challenge becomes a true statement only when n is replaced by 1.

2. Have pupils recall that the product of a given number and 1 is the given number. The number 1 is called the identity element of multiplication.

B. Review multiplicative inverses

1. By what numeral should n be replaced if each of the following is to be a true statement?

   $2n = 1$ $(\frac{1}{2})$ $\qquad$ $\frac{4}{5}n = 1$ $(\frac{5}{4})$

   $\frac{1}{3}n = 1$ (3) $\qquad$ $-3n = 1$ $(-\frac{1}{3})$

2. Elicit that two numbers whose product is 1 are called multiplicative inverses (reciprocals). Each is the multiplicative inverse of the other.

3. a. What is the multiplicative inverse of 2? Why?
   What is the multiplicative inverse of $\frac{1}{3}$? Why?

   b. Write the multiplicative inverse of each of the following:

   5 $\qquad$ -2 $\qquad$ $\frac{1}{8}$ $\qquad$ $-\frac{4}{9}$ $\qquad$ $\frac{7}{4}$

4. Write another name for each of the following products:

$3\,(\frac{1}{3})$ $\quad$ $-5\,(-\frac{1}{5})$

$\frac{1}{4}\,(4)$ $\quad$ $(-\frac{2}{7})\,(-\frac{7}{2})$

5. Write another name for each of the following products:

$3\,(\frac{1}{3}x)$ (x because $(3 \times \frac{1}{3})x = 1x$ or x) $\quad$ $-5\,(-\frac{1}{5}b)$

$\frac{1}{4}\,(4y)$ $\quad$ $-\frac{2}{7}\,(-\frac{7}{2}r)$

6. By what do we multiply each expression in Column A to obtain the corresponding expression in Column B.

A	B
$4n$	$n\,(\frac{1}{4})$
$-10x$	$x\,(-\frac{1}{10})$
$\frac{1}{2}r$	r
$\frac{3}{7}y$	y
$\frac{1}{5}a$	a

Note: Pupils should be able to justify their answers as follows:

$\frac{1}{4}\,(4n)$

$= (\frac{1}{4} \times 4)n$ Associative principle of multiplication

$= 1n$ Multiplicative inverse

$= n$ Multiplicative identity

C. To solve and check equations of the type $ax = b$ (a, b are rational numbers and $a \neq 0$) using the multiplication principle

1. What number has been used to multiply both members of each equation in column A to obtain the corresponding equation in column B?

A	B
$3x = 6$	$x = 2\,(\frac{1}{3})$
$4y = 16$	$y = 4\,(\frac{1}{4})$
$-7y = 35$	$y = -5\,(-\frac{1}{7})$
$\frac{1}{2}a = 9$	$a = 18\,(2)$

How are the equations in each pair related? (They are equivalent.)
How can you tell?
Of the two equations in each pair, which is the easier to solve?
Why?

2. Consider the equation $4x = 8$. How can we find the equivalent equation whose solution is obvious?

   a. By what would you multiply the left and right members of the equation to have x appear alone as one member? ($\frac{1}{4}$)

   b. Have pupils complete the solution of $4x = 8$ as follows:

$$\begin{aligned} 4x &= 8 \\ \tfrac{1}{4}(4x) &= \tfrac{1}{4}\ (8) \\ x &= 2 \end{aligned}$$

   c. How would you check the solution of $4x = 8$?

3. Consider the equation: $\frac{x}{5} = 2$.

   a. Have pupils recall that if we multiply by the reciprocal (multiplicative inverse) of a number, the result is the same as if we had divided by the number.

   b. Have pupils express: $\frac{x}{5} = 2$ as $\frac{1}{5}x = 2$ and complete the solution as follows:

   What is the multiplicative inverse of $\frac{1}{5}$?

$$\begin{aligned} \tfrac{1}{5}x &= 2 \\ 5(\tfrac{1}{5}x) &= 5\ (2) \\ (5 \times \tfrac{1}{5})x &= 5\ (2) \\ x &= 10 \end{aligned}$$

   c. How would you check the solution of $\frac{1}{5}x = 2$?

II. Practice

A. By what number would you multiply both members of each equation to obtain the equivalent equation whose solution is obvious?

1. $10a = 50$
2. $24 = 3a$
3. $-2x = 12$
4. $\frac{1}{4}y = 6$
5. $2.5b = 10$
6. $4r = -12$
7. $\frac{2}{3}t = 4$
8. $-8b = -1.6$
9. $4.2 = 2.1y$
10. $-2\frac{1}{2} = \frac{x}{2}$

B. Solve and check each of the equations in A.

III. Summary

A. What is the identity element of multiplication in the set of rationals?

B. What is meant by the multiplicative inverse of a number? the reciprocal of a number?

C. In equations such as those we have studied today, how can you tell by what number both members of the equation should be multiplied to obtain an equation whose solution is obvious?

D. How can you be sure that the solution set of the "obvious" equation is also the solution set of the given equation?

Lesson 71

Topic: Verbal problems

Aim: To solve verbal problems

Specific Objectives:

To express English sentences using mathematical symbols
To review steps in analyzing verbal problems
To solve verbal problems using equations of the type $ax = b$ or $x \pm a = b$

Challenge: In a certain school $\frac{2}{5}$ of the students of the 8th grade come by bus. If 100 students ride by bus to school, how many students are in the 8th grade?

I. Procedure

A. Expressing English phrases and sentences using mathematical symbols

1. How could we represent each of the following?

   a. Five times a certain number ($5\square$, $5x$, $5n$, $5a$)

   b. A number increased by 7

   c. A number decreased by 10

   d. $\frac{1}{3}$ of a number

2. If we represent the number by x, how can we express each of the following sentences in mathematical symbols?

   a. Nine times a certain number is 63. ($9x = 63$)

   b. A number increased by four is equal to 17. ($n + 4 = 17$)

B. To review steps in analyzing a verbal problem

1. Refer to challenge. What does the problem ask us to find? (the number of students in the 8th grade)

2. How can we represent the number of students in the 8th grade? (n)

3. How can we represent $\frac{2}{5}$ of that number in terms of n? ($\frac{2}{5}n$)

4. How can we express the conditions of the problems in the form of an open sentence? ($\frac{2}{5}n = 100$)

5. Estimate: Is the number greater or less than 100? Why?

C. Solving problems

1. Solve the equation

$$\frac{2}{5}n = 100$$
$$\frac{5}{2}\left(\frac{2}{5}n\right) = \frac{5}{2}\ (100)$$
$$\left(\frac{5}{2}\cdot\frac{2}{5}\right)n = \frac{5}{2}\ (100)$$
$$n = 250$$

If there are 250 students in the 8th grade, will $\frac{2}{5}$ of that number equal 100?

How do we check to discover if our solution is correct?

Note to Teacher: Lead pupils to see that the answer to a word problem should not be checked in the equation but in the words of the problem. If the equation set up by the pupil does not correctly depict the conditions of the problem, yet it has been solved correctly, the solution will check the equation but not be a solution for the problem.

2. In a class election, Alice received 11 more votes than Jean received. If Alice received 31 votes, how many did Jean receive?

Guide the pupils to use a procedure such as the following:

Estimate: Jean received less than 31 votes.

Let n = number of votes Jean received

then n + 11 = number of votes Alice received

$$n + 11 = 31$$
$$n + 11 + (-11) = 31 + (-11)$$
$$n = 20$$

The number of votes received by Jean is 20.

Check: Is 31 eleven more than 20?

3. John spent $6 from his weekly wages. He had $38 left. How much did he earn in a week?

Estimate: More than $38.

Let n = number of dollars earned in a week
then n - 6 = number of dollars left

$$n - 6 = 38$$
$$n - 6 + (6) = 38 + (6)$$
$$n = 44$$

John earned $44 per week.

Check: If you spend \$6 out of \$44, will you have \$38 left?

II. Practice

A. Express each of the following as a mathematical sentence:

1. A number increased by 5 is 12.

2. Five times a number is equal to 9.

3. A number increased by 7 is equal to 14.

B. A family on an auto trip drives 400 miles per day. How many days will it take them to travel a distance of 1600 miles?

C. During spring training, a baseball shortstop lost $11\frac{3}{4}$ pounds. His weight then was $183\frac{1}{2}$ pounds. How much did he weigh before spring training?

D. In a mathematics test, Sam answered 24 questions correctly and received a mark of 80%. How many examples were on the test? (Remind pupils that 80% may be written as $\frac{8}{10}$ or $\frac{4}{5}$.)

III. Summary

A. What are the steps in analyzing a problem?

B. What procedure do we follow in order to solve a problem? (Analyze, form an equation, solve)

C. What mathematical principles did you use in solving problems?

Lesson 72

Topic: Solving equations

Aim: To learn to solve equations using two properties of equality

Specific Objectives:

To observe that one operation alone may not be sufficient to solve an equation

To transform an equation of the form $ax \pm b = c$ when $a \neq 0$ into an equivalent equation whose solution is obvious

Challenge: How would you transform $3x + 4 = 6$ to an equivalent equation whose solution is obvious?

I. Procedure

A. One operation may or may not be sufficient to solve an equation

1. Solve the following, first telling what principle of equation solving you will use:

    a. $3x = 8$

    b. $x + 3 = 8$

    Elicit that in each case only one operation is used to transform the given equation into an equivalent equation whose solution is obvious.

2. Translate into mathematical symbols the following English sentence: Twice a number increased by five equals fourteen.

    a. Translation: $2n + 5 = 14$

    b. What equivalent simple equation would be obtained by use of the addition property of equality?

    $2n + 5 + (-5) = 14 + (-5)$ We added (-5) to each member.
    $2n + 0 = 9$
    $2n = 9$

    Explain why this is not an equivalent equation with an obvious solution.

    c. In the equation $2n + 5 = 14$, what equivalent equation do we obtain by use of the multiplication property of equality?

    $2n + 5 = 14$
    $\frac{1}{2}(2n+5) = \frac{1}{2}(14)$ We multiplied each member by $\frac{1}{2}$.
    $n+2\frac{1}{2} = 7$

    Is this an equivalent equation with an obvious solution? Explain.

d. Therefore, we conclude that an equation cannot always be solved in one step.

B. Using two operations

1. Carrying the equations $2n = 9$ and $n + 2\frac{1}{2} = 7$ each a step further, we have:

   a. $2n = 9$
      $n = 4\frac{1}{2}$ Multiplication property

   b. $n + 2\frac{1}{2} = 7$
      $n = 4\frac{1}{2}$ Addition property

   In each case, the solution set of $2n + 5 = 14$ is $\{4\frac{1}{2}\}$.

2. Elicit that we used two properties of equality to solve the equation $2n + 5 = 14$.

3. Returning to the challenge:

   a. If we use the multiplication property first, and then the addition property, our example would look like this:

$$
\begin{aligned}
3x + 4 &= 16 \\
\tfrac{1}{3}(3x+4) &= \tfrac{1}{3}(16) \quad \text{Multiplication property} \\
x + \tfrac{4}{3} &= \tfrac{16}{3} \\
x + \tfrac{4}{3} + (-\tfrac{4}{3}) &= \tfrac{16}{3} + (-\tfrac{4}{3}) \quad \text{Addition property} \\
x &= \tfrac{12}{3} \\
\text{or} \qquad x &= 4 \quad (\text{The solution set of } 3x + 4 = 16 \text{ is } \{4\})
\end{aligned}
$$

   b. If we use the addition property first, and then the multiplication property, our example would look like this:

$$
\begin{aligned}
3x + 4 &= 16 \\
3x+4+(-4) &= 16 + (-4) \quad \text{Addition property} \\
3x &= 12 \\
\tfrac{1}{3}(3x) &= \tfrac{1}{3}(12) \quad \text{Multiplication property} \\
x &= 4
\end{aligned}
$$

   c. Elicit that in each case, the solution set of $3x + 4 = 16$ is $\{4\}$.

   d. Which of these procedures seems easier? Elicit that using the addition property first, and then the multiplication property is the more convenient procedure.

   e. Have pupils check the solution in the original equation.

II. Practice

A. By using the addition property first, and then the multiplication property, solve and check the following equations:

1. $4x + 7 = 11$
2. $5n + 4 = -21$
3. $11y + 12 = 89$
4. $2x - 6 = 7$
5. $7a - 11 = -18$
6. $3y - 3 = 3$

B. Write an equation in which two properties of equality will be used to find the solution set.

III. Summary

A. Why is the use of just one property of equality not always adequate to solve an equation?

B. In solving an equation such as $3x - 7 = 20$, how many operations are needed? Which would you use first?

C. If you changed the order in which you used the operations needed to solve a two-step equation, how would the results compare? Explain.

Lesson 73

Topic: Verbal Problems

Aim: To solve verbal problems which lead to equations whose solution requires the use of two properties of equality

Specific Objectives:

To review the translation of English phrases and sentences into mathematical symbols
To review steps in analyzing a problem
To solve problems leading to an equation of the type: $ax \pm b = c$

Challenge: Twice a number increased by five is thirty-seven. What is the number?

I. Procedure

A. Review the translation of English phrases and sentences into mathematical symbols

1. Write each of the following in mathematical symbols. In each case, tell what the variable represents.

   Note: Be sure pupils understand that the variable represents a number.

   a. The bicycle costs seven dollars more than the amount I have.
   b. The sum of a number and six is eleven.
   c. The length of a rectangle is five inches more than its width.
   d. Michael is two inches taller than Ben.
   e. 6 more than 4 times a number

2. Which of the above are sentences and which are phrases? Explain the difference.

B. Review steps in analyzing a problem
Refer to challenge.

1. What does the problem ask us to find?

2. How can we represent the number? (n)

3. How can we represent "twice a number increased by 5" in terms of n?

4. How can we express the conditions of the problem in the form of an open sentence? ($2n+5=37$)

C. Solving problems

1. Guide pupils to use the following procedure:

$$\begin{aligned} \text{Let } n &= \text{the number} \\ 2n+5 &= 37 \\ 2n+5+(-5) &= 37 + (-5) \\ 2n &= 32 \\ n &= 16 \end{aligned}$$

The number is 16.

2. How do we check to find out if our solution is correct? Is the sum of twice the number and 5 equal to 37? Yes.

II. Practice

A. In a class election, Alice received 11 more votes than twice the number that Jean received. If Alice received 31 votes, how many did Jean receive?

B. The length of a room is seven feet less than three times the width. If the length is twenty-six feet, determine the width of the room.

C. Bob's age is eight years more than twice Serita's. If Bob is thirty-two years old, how old is Serita?

III. Summary

A. Explain the steps we follow to analyze a verbal problem.

B. What properties of equality did you use in solving the problems in today's lesson?

Lesson 74

Topic: Proportion

Aim: To review the concept of ratio

Specific Objectives:

To review the concept of ratio as an ordered pair of numbers
To review expressing a ratio in simplest form
To review the use of ratios to compare measures

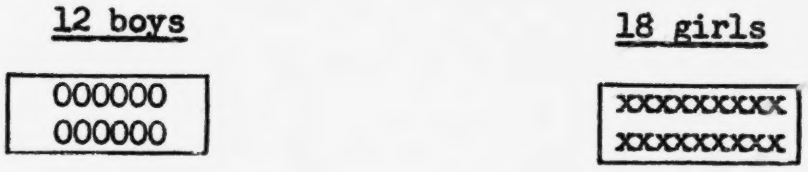
Challenge: What is the simplest name for: $\frac{30}{50}$, $\frac{6}{10}$, $\frac{15}{25}$, $\frac{3}{5}$?

I. Procedure

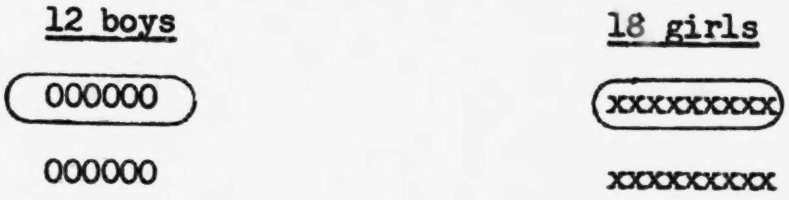
A. Review concept of ratio

1. In an 8th grade class, 12 boys and 18 girls joined the G.O. How could we express the comparison of the number of boys to the number of girls?

2. To compare the number of boys and the number of girls who joined the G.O., we might represent the set of boys and the set of girls as follows:



3. The comparison of the number of boys to the number of girls may be expressed as 12 to 18. Are there any other ways of expressing this comparison?



For every six boys who joined, nine girls joined, or the number of boys compared to the number of girls may be expressed as 6 to 9.

4. Elicit other comparisons such as 4 boys to 6 girls or 2 boys to 3 girls.

Have the pupils realize that for every 2 boys who joined the G.O., there were 3 girls who joined the G.O.

We can express the ratio of the number of boys to the number of girls in several ways: 12 to 18, 6 to 9, 4 to 6, and 2 to 3. We say that all of these are equivalent ratios.

5. Recall that the order of the numbers is important. Since we are comparing the number of boys to the number of girls, the ratio is 2 to 3. If we were comparing the number of girls to the number of boys, the ratio would be 3 to 2. For this reason, a ratio is said to consist of an ordered pair of numbers.

6. Have pupils recall the various ways to express a ratio. For example, the ratio of 5 to 8 may be expressed as 5:8 or $\frac{5}{8}$. (Read: 5 to 8.)

   Have pupils express the following ratios in various ways: 8 to 12; 13 to 21; 25 to 10; 7 to 9.

B. Review expressing a given ratio in its simplest form

Refer to challenge.

1. Have pupils recall that a ratio is named in simplest form when its terms are relatively prime, i.e., they have no common factor other than 1.

   The simplest form for the name of each of the ratios in the challenge is: $\frac{3}{5}$. (Read: "3 to 5.")

2. Have pupils recall that the multiplicative identity, 1, may be used to find equivalent names for a given ratio.

a. $\frac{6}{10} = \frac{2 \times 3}{2 \times 5}$

$= \frac{2}{2} \times \frac{3}{5}$

$= 1 \times \frac{3}{5}$

$= \frac{3}{5}$

$\frac{6}{10}$ and $\frac{3}{5}$ name the same ratio. Therefore, they are called equivalent ratios.

b. $\frac{2}{3} = 1 \times \frac{2}{3}$

$= \frac{4}{4} \times \frac{2}{3}$

$= \frac{4 \times 2}{4 \times 3}$

$= \frac{8}{12}$

$\frac{2}{3}$ and $\frac{8}{12}$ are equivalent.

c. Find an equivalent ratio for each of the following:

$\frac{1}{3}$, $\frac{1}{2}$, $\frac{4}{6}$, $\frac{3}{4}$, $\frac{14}{16}$ (an equivalent ratio for 1 to 3 is 2 to 6, etc.)

3. Express the following ratios in simplest form: $\frac{9}{15}$, $\frac{12}{18}$, $\frac{18}{27}$, $\frac{35}{45}$.

C. To review the use of ratios to compare measures

1. Using ratios to compare like measures

a. Have pupils recall that if a ratio is used to compare two like measures, i.e., two linear measures, two measures of weight, etc., to avoid confusion it is customary to use the same unit of measure for both.

b. Express the ratio in simplest form.

5 inches to 1 foot (5" to 12" or 5:12)
1 hour to 20 minutes
8 ounces to 2 pounds

Note to Teacher: In scale drawing or map reading, we do not follow the procedure of changing to the same unit. For example, on a map it is more convenient to know the scale is 1 inch = 200 miles (read 1 inch represents 200 miles) rather than a ratio of 1:12,672,000.

2. Introducing the concept of rate

a. Have pupils consider the following:

35 miles per hour
32 feet per second
5 inches every 20 minutes
4 pens for a dollar

Elicit that in each of the above situations unlike measures are being compared. In "35 miles per hour" a measure of distance is compared to a measure of time. In such situations, there is no common unit of measure. However, it is possible to compare the number of miles to the number of hours.

b. Mr. Jones drove 160 miles in 4 hours. The ratio of the number of miles to the number of hours is $\frac{160}{4}$ or $\frac{40}{1}$. We usually say he drove at the rate of 40 miles per hour. A ratio involves only numbers and no units. The rate is the number of miles per hour.

c. Tell pupils we do not express a rate in fractional form. We usually use the words per, for, or every to state the relationship between the two measures as: 60 miles per hour; $4 for 3 records, etc.

II. Practice

A. The committee for a G.O. dance consisted of 3 boys and 5 girls

1. What is the ratio of the number of boys to the number of girls on the committee?

2. What is the ratio of the number of girls to the number of boys on the committee?

3. What is the ratio of the number of boys to the total number of committee members?

4. What is the ratio of the number of girls to the total number of committee members?

B. Express the following ratios in simplest form:

1. 2 to 4
2. 15 to 25
3. 18 to 19
4. 21 to 37

C. Express each of the following as a ratio in simplest form:

1. 5 yards to 4 feet
2. 2 lbs. to 12 oz.
3. 1 hour to 15 minutes

D. Find the rate in simplest form for each of the following pairs of measure:

1. 90 miles in 3 hours (30 miles per hour)

2. 8 pints for $2

3. 3261 feet in 3 seconds

III. Summary

A. In what way is a ratio an example of an ordered pair of numbers?
B. When is a ratio said to be in simplest form?
C. How is the multiplicative identity used to write a ratio equivalent to a given ratio?
D. Explain why in expressing a rate we must state the two measures involved.

Lesson 75

Topic: Proportion

Aim: To develop the concept of a proportion

Specific Objectives:

To develop the concept of proportion
To learn the vocabulary used in expressing a proportion
To learn that in a proportion the product of the means is equal to the product of the extremes

Challenge: Is the following statement true or false? 38:57 = 30:40

I. Procedure

A. Concept of proportion

1. Is the ratio $\frac{3}{5}$ equivalent to the ratio $\frac{9}{15}$? We may then write the sentence: $\frac{3}{5} = \frac{9}{15}$. A sentence which says that two ratios are equivalent is called a proportion. This proportion is read: "3 is to 5 as 9 is to 15."

   Note: The proportion might have been written 3:5 = 9:15.

2. Elicit that a sentence comparing two ratios is not read in the same way as a sentence comparing two fractions.

   If $\frac{3}{5}$ and $\frac{9}{15}$ represent fractional numbers, the statement $\frac{3}{5} = \frac{9}{15}$ would be read: "three fifths is equivalent to nine fifteenths."

3. Complete the following proportions:

   $\frac{3}{4} = \frac{?}{8}$ $\frac{7}{8} = \frac{14}{?}$

   $\frac{?}{100} = \frac{3}{20}$ $\frac{2}{?} = \frac{4}{12}$

4. Have pupils write each proportion in #3 in another way, e.g. 3:4 = 6:8.

5. Have pupils read the completed proportions in #4.

B. Vocabulary used in proportion

1. Complete the following proportion: ☐ : 15 = 12:45.

2. Tell the pupils that in the proportion 4:15 = 12:45, 4 is called the first term, 15 the second term, 12 the third term, and 45 the fourth term.

The first and fourth terms are called the extremes and the second and third terms are called the means.

extremes
$$4:15 = 12:45$$
means

3. If the proportion is written $\frac{4}{15} = \frac{12}{45}$, which are the extremes? (4, 45) Which are the means? (15, 12)

4. In proportions a and b, name the means. In proportions c and d, name the extremes.

   a. $70:10 = 35:5$

   b. $\frac{14}{56} = \frac{1}{4}$

   c. $\frac{3}{8} = \frac{21}{56}$

   d. $2:5 = 10:25$

C. In a proportion, the product of the means is equal to the product of the extremes

1. Have pupils recall that a proportion is defined as a sentence which says that two ratios are equivalent.

2. By writing the ratios in simplest form, decide which of the following are proportions.

   a. $\frac{1}{4} = \frac{3}{12}$

   b. $\frac{3}{8} = \frac{6}{16}$

   c. $\frac{2}{3} = \frac{4}{9}$

   d. $3:11 = 9:33$

   e. $4:5 = 2:3$

   f. $6:12 = 1:2$

3. Refer to each of the problems above to answer the following:

   a. What is the product of the means?
   b. What is the product of the extremes?
   c. How does the product of the means compare with the product of the extremes, when the statement is true? when the statement is false?

4. Elicit that in a proportion the product of the means is equal to the product of the extremes.

   Refer to the challenge: $38:57 = 30:40$.

   Does $57 \times 30 \overset{?}{=} 38 \times 40$

   $$1710 \overset{?}{=} 1520$$

   $$1710 \neq 1520$$

Since 1710 does not equal 1520, the statement in the challenge is false and therefore it is not a proportion.

This procedure for testing whether a statement is a proportion is called the "products test."

5. Using the products test, tell which of the following are proportions.

a. $10:2\frac{1}{2} = 60:13$

b. $\frac{1}{2}:\frac{1}{3} = 6:4$

c. $7:9 = 14:15$

d. $8:3 = 56:21$

II. Practice

A. Complete the following proportions so that each results in a true statement:

1. $\frac{1}{2} = \frac{?}{16}$

2. $\frac{3}{4} = \frac{15}{?}$

3. $\frac{?}{20} = \frac{3}{10}$

B. Name the means and extremes in the following proportion: $6:9 = 10:15$.

C. Interchange the means and the extremes. Write the proportion.

D. Using the products test, find which of the following statements are proportions.

$9:16 = 36:64$

$6:10 = 21:28$

$48:20 = 12:5$

$5:8 = 8:5$

III. Summary

A. What is a proportion?

B. In a proportion, which terms are the means and which are the extremes?

C. What is meant by the "products test" to determine whether a statement is a proportion?

D. What new vocabulary have you learned today?

(proportion, means, extremes)

Lesson 76

Topic: Proportion

Aim: To use proportions in problem solving

Specific Objectives:

To learn to solve proportions which are open sentences
To solve problems involving proportions

Challenge: What value for x will make this open sentence a proportion?

$$\frac{x}{25} = \frac{3}{15}$$

I. Procedure

A. Solving a proportion

1. Use the products test to tell whether the following statements are proportions.

a. $\frac{2}{3} = \frac{8}{12}$ $(3\times 8 = 2\times 12)$	a proportion
b. $\frac{3}{5} = \frac{9}{10}$ $(5\times 9 \neq 3\times 10)$	not a proportion
c. $\frac{3}{4} = \frac{n}{20}$ $(4n = 3\times 20)$	cannot tell

2. Consider example c above.

a. Elicit that $\frac{3}{4} = \frac{n}{20}$ is an open sentence.

What value for n will result in a proportion?

Elicit that if we have a proportion, $4n = 3\times 20$ (products test)
$4n = 60$

Solving for n we obtain: $n = 15$.

b. Replacing n by 15 in the open sentence, we have $\frac{3}{4} = \frac{15}{20}$.
Is the ratio of 3 to 4 equivalent to the ratio of 15 to 20?

c. When we find the replacement for the variable which results in a proportion, we say we have solved the proportion.

3. Refer to challenge.

a. Have pupils attempt to solve the proportion by equivalent ratios.

b. Suggest that by using the principle that in a proportion "the product of the means equals the product of the extremes," perhaps the problem will be easier to solve.

$$\frac{x}{25} = \frac{3}{15}$$

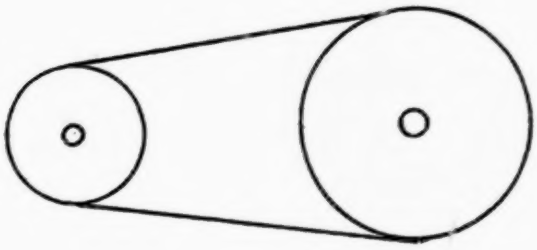
$$15x = 75$$

$$x = 5$$

c. Check. Is the ratio of 5 to 25 equivalent to the ratio of 3 to 15?

B. Using proportion to solve problems

1. In a certain machine a small wheel and a large wheel are connected by a belt.



The small wheel makes 9 turns for every 4 turns made by the large wheel. How many turns does the small wheel make when the large wheel makes 238 turns?

Note to Teacher: Review steps in analyzing a verbal problem.

a. What does the problem ask us to find? (The number of turns made by small wheel while the large wheel makes 288 turns.)

b. How can we represent this number? (n)

c. How can we express the ratio of 9 turns of the small wheel to 4 turns of the large wheel?

Number of turns - small wheel $\frac{9}{4}$
Number of turns - large wheel

d. How can we express in the same order the ratio of n turns of the small wheel to 288 turns of the large wheel?

Number of turns - small wheel $\frac{n}{288}$
Number of turns - large wheel

e. Express the conditions of the problem as an open sentence.

$$\frac{9}{4} = \frac{n}{288}$$

f. Solve the proportion and check the answer.

2. If three cans of tomato paste cost 25 cents, how much will 12 cans of tomato paste cost?

   a. What does the problem ask us to find?

   b. How can we represent the number of cents that a dozen cans will cost? (n)

   c. How can we express the ratio of the number of cans to the number of cents?

      $$\frac{\text{number of cans}}{\text{number of cents}} \quad \frac{3}{25}$$

   d. How can we express in the same order the ratio of 12 cans to n cents?

      $$\frac{\text{number of cans}}{\text{number of cents}} \quad \frac{12}{n}$$

   e. How can we express the conditions of the problem in the form of an open sentence?

      $$\frac{3}{25} = \frac{12}{n}$$

   f. Solve the proportion:

      $$\begin{aligned} 12 \times 25 &= 3n \\ 300 &= 3n \\ 100 &= n \end{aligned}$$

   g. Check.

3. On a map, 1 inch represents a distance of 10 miles. Two towns are 4 inches apart on the map. What is the actual distance between the towns? (Let n represent the number of miles in the actual distance.)

   a. $$\frac{\text{number of inches on map}}{\text{number of miles on ground}} \quad \frac{1}{10}$$

   b. $$\frac{\text{number of inches on map}}{\text{number of miles on ground}} \quad \frac{4}{n}$$

   c. Express the conditions of the problem as an open sentence.

      $$\frac{1}{10} = \frac{4}{n}$$

   d. Solve the proportion.

   e. What is the actual distance in miles between the towns? Does your answer check?

II. Practice

A. Use the products test to tell whether the following are or are not proportions.

1. $\frac{6}{7} = \frac{54}{63}$ 2. $4:12 = 24:8$ 3. $\frac{3}{15} = \frac{6}{12}$

B. For which value of n is each of the following a proportion?

1. $\frac{1}{3} = \frac{2}{n}$ 2. $\frac{4}{n} = \frac{16}{20}$ 3. $\frac{n}{11} = \frac{6}{22}$

C. Use proportions to solve the following problems:

1. In a school, the ratio of the number of eighth grade pupils to the total number of pupils is 2 to 3. If the total register of the school is 2400 pupils, how many eighth grade pupils are there?

2. On a road map 1" represents 45 miles. Using the same scale, how many inches will represent 135 miles?

III. Summary

A. What rule is used to solve a proportion which is an open sentence?

B. Show by giving an illustration how you would find one term of a proportion if you knew the other three.

C. What procedure would one follow to solve problems using proportion?

Lessons 77 and 78

Topic: Proportion

Aim: To use proportion to solve problems involving per cent

Specific Objectives:

To review per cent expressed as a ratio of a number to the number 100
Solving problems involving per cent through the use of proportion

Challenge: In a test, Sally had 17 examples correct. If the test had 20 examples, what per cent of the examples did Sally have correct?

I. Procedure

A. To review per cent expressed as a ratio

1. Have pupils recall that per cent can be considered as the ratio of a number to the number 100.

75% means the ratio of 75 to 100 or $\frac{75}{100}$

80% means the ratio of $\square$ to 100 or $\frac{\square}{100}$

65% means the ratio of 65 to $\square$ or $\frac{65}{\square}$

32% means the ratio of Δ to $\square$ or $\frac{\Delta}{\square}$

x% means the ratio of x to $\square$ or $\frac{x}{\square}$

2. Elicit that the ratio $\frac{11}{100}$ may be written as 11%.

Write each ratio as a per cent: $\frac{17}{100}$, $\frac{100}{100}$, $\frac{135}{100}$, $\frac{2}{100}$, $\frac{n}{100}$.

B. Using proportion to solve problems involving per cent

Note: The following problems have been chosen to illustrate the use of proportion in solving the three basic problem situations involving per cent.

1. Finding what % one number is of another
Refer to challenge.

a. What do we wish to find? (The per cent of examples Sally had correct.)

b. How may we represent this per cent? (x%)

c. Represent x% as a ratio of x to 100 ($\frac{x}{100}$).

Elicit that since Sally did not have all the examples correct, she will receive less than 100%. Therefore, we know $x < 100$.

d. What is the ratio of the number of examples Sally had correct to the number of examples on the test? ($\frac{17}{20}$).

e. Remind pupils that ratios can be considered as ordered pairs, i.e., the order in which the terms appear is important. In setting up a proportion, care must be taken to see that the order of the terms of each of the two equivalent ratios agree.

f. Help pupils to set up the proportion: $\frac{17}{20} = \frac{x}{100}$.

Guide pupils to see that 17 (number of correct examples) has the same relationship to 20 (total number of examples) as x has to 100.

g. What method may we use to find the value of x? (The product of the means equals the product of the extremes.)

$$20x = 1700$$
$$x = 85$$

In the original proportion, if we replace x with 85, $\frac{x}{100}$ becomes $\frac{85}{100}$ and the answer is 85%.

h. Check. Is the ratio of 17 to 20 equivalent to the ratio of 85 to 100?

i. In a similar way, find what per cent 27 is of 50; 14 is of 28.

2. Finding a per cent of a given number

In a group of 300 pupils, 37% had received an "A" in mathematics. How many received an "A"?

a. What are we asked to find? (the number of "A" pupils)

b. How will we represent this number? (x)

c. Elicit that since $37\% < 100\%$, $x < 300$.

d. How will we represent 37% as a ratio? ($\frac{37}{100}$)

e. What ratio will express the comparison between the number of "A" pupils and the number of pupils in the group? ($\frac{x}{300}$)

f. Have pupils express the proportion ($\frac{x}{300} = \frac{37}{100}$).

g. Elicit that the number of "A" pupils has the same relationship to the total number of pupils as 37 has to 100.

h. Solve the proportion

$$\frac{x}{300} = \frac{37}{100}$$

$$100x = 11100$$

$$x = 111$$

Answer: 111 pupils received "A".

i. Check. Is the ratio of 111 to 300 equivalent to the ratio of 37 to 100?

j. In a similar way, find 80% of 120; 35% of 140.

3. Finding a number when a per cent of it is known

A boy sold 24 tickets for a dance. This was 4% of the tickets sold. How many tickets were sold?

a. What are we asked to find? (the number of tickets sold)

b. How will we represent this number? (x)

c. Elicit that 24 represents only a part (4%) of the total number of tickets sold, and that, therefore, x (the number of tickets sold) is greater than 24.

d. Have pupils set up the proportion showing that 24 has the same relationship to x as 4 has to 100.

e. Have pupils solve the proportion

$$\frac{24}{x} = \frac{4}{100}$$

$$2400 = 4x$$

$$600 = x$$

Answer: 600 tickets were sold.

f. Check. Is the ratio of 24 to 600 equivalent to the ratio of 4 to 100?

II. Practice

A. A basketball player is successful in shooting 13 fouls out of 20 tries. What per cent of shots were successful?

B. A pupil gets 23 examples correct on a test which has 25 questions. What per cent did he get right?

C. There are 250 pupils in an auditorium. 40% of them are girls. How many girls are there in the auditorium?

D. The traffic during a holiday weekend was 150% of regular weekend traffic. If there are 24 million cars on the road during a regular weekend, how many cars are there on the road during a holiday weekend?

E. A company pays 7% of the yearly salary to each employee as a bonus at the end of the year. Mr. Jones received $560 as a bonus. What is Mr. Jones' yearly salary?

F. 40% of the eighth grade pupils joined the G.O. during the first week of school. If this represents 80 pupils, what is the register of the eighth grade?

III. Summary

A. In expressing a per cent as a ratio, what is the second term?

B. What proportions can be used to find:

1. What per cent of 52 is 13?

2. What is 6% of 720?

3. The number of which 5 is 20%?

CHAPTER VIII

THE SET OF IRRATIONAL NUMBERS

This chapter suggests procedures for introducing pupils to the set of real numbers. Methods are presented for developing important basic understandings of the meaning of irrational numbers and how they relate to rational numbers in the set of real numbers.

The materials included in this section will help to develop pupil understanding of:

- squares of rational numbers (perfect squares)
- square root; principal square root symbol: $\sqrt{\ }$; radicand
- solution of problems which involve finding the square root of a perfect square
- rational numbers - terminating or repeating decimals
- irrational numbers - non-terminating, non-repeating decimals
- square root of a non-perfect square by approximation
- irrational numbers and the number line

As the pupil reviews the squaring of a rational number and practices squaring various elements of the set of rational numbers, he is guided to consider the inverse operation, i.e., finding the square root of a number. The meaning of the radical sign, "$\sqrt{\ }$" is developed, along with the distinction between the square roots of a number and its principal square root. An estimation and trial method of finding square roots (of perfect squares) is suggested as appropriate to this stage of pupil understanding.

The approach to the concept of an irrational number is by means of an investigation of decimals. Pupils find that any rational number $\frac{a}{b}$ $(b \neq 0)$ can be expressed as a decimal that either terminates or repeats. Conversely, any terminating or repeating decimal represents a rational number. The question is then raised of a non-terminating, non-repeating decimal, such as: .6325447... Since non-repeating, non-terminating decimals cannot represent rational numbers, we give them the name irrational numbers. Then the set of real numbers is identified as the set consisting of all rational numbers and all irrational numbers.

In this chapter, pupils are guided to see that the square root of every positive rational number cannot always be found. They are helped to understand that the square roots of some rational numbers are irrational. Procedures are presented for having pupils develop a technique for finding a rational approximation (to tenths place) of a non-perfect square.

An approximation method of associating a point on the number line with an irrational number is also developed. Thus, pupils are guided to an understanding of the real number line.

Through a study of right triangles, pupils discover the Pythagorean relationship and learn to express this relationship in the generalized form: $c^2 = a^2 + b^2$. They are then introduced to a method of solving problems involving this Pythagorean relationship. In the course of solving such problems, the use of the square root tables in the text is introduced to facilitate computation.

CHAPTER VIII

THE SET OF IRRATIONAL NUMBERS

Lessons 79-88

Lessons 79 and 80

Topic: Square Root

Aim: To learn the meaning of square root

Specific Objectives:

To introduce the concept of a perfect square
To introduce the concept of square root
To learn the use of the symbol: $\sqrt{\quad}$

Challenge: What have these numbers in common? 81, $\frac{4}{9}$, 49, and .25?

I. Procedure

A. The concept of a perfect square

1. Have pupils recall that 3×3 can be written as 3^2.

   a. In the expression 3^2, what is the exponent? What is the base? What does the exponent indicate?

      Have pupils recall that an expression such as 3^2 is called a power and read "the second power of 3."

   b. How can we write the second power of 4? of 5? In the expression 5^2, how many times is 5 used as a factor?

2. Have pupils recall that 3^2 can also be read as "3 square" or "3 squared." When we square a number, we use the number as a factor twice.

3. Meaning of perfect square

   a. Have pupils complete the following:

$9^2 = ?$	$(9 \times 9 = ?)$
$(\frac{2}{3})^2 = ?$	$(\frac{2}{3} \times \frac{2}{3} = ?)$
$(-7)^2 = ?$	$(-7 \times -7 = ?)$
$(-.5)^2 = ?$	$(-.5 \times .5 = ?)$
$0^2 = ?$	$(0 \times 0) = ?$

b. To what set of numbers do the following belong?
$9, \frac{2}{3}, -7, -.5, 0$? (the set of rational numbers)

c. Tell pupils that when we square a rational number, the result, always a rational number, is called a perfect square.

d. Refer to the challenge. What the numbers 81, $\frac{4}{9}$, 49, and .25 have in common is that each is a perfect square. Each is the square of a rational number.

4. If a positive number is squared, what is the sign of the result? (positive)
If a negative number is squared, what is the sign of the result? (positive)
Elicit that perfect squares, except zero, are positive rational numbers. (Zero is considered to be neither positive nor negative.)

B. The concept of square root

1. a. Name some pairs of factors of 36. (1x36, 2x18, 4x9, 6x6)

b. Is there a pair of factors where each factor is the same number?

c. Since 6x6 = 36, 6 is called a square root of 36. In the same way, since 5x5 = 25, 5 is called a square root of 25.

Why is 7 a square root of 49?
Why is 9 a square root of 81?
Why is b a square root of b^2?

2. a. Have pupils make a table of squares: $1^2,\ldots,10^2$.

I number (n)		II number squared (n^2)
1	1x1	1
2	2x2	4
3	3x3	9
.	.	.
.	.	.
.	.	.
10	10x10	100

Elicit that each number in column II is a perfect square, i.e., the square of the corresponding number in column I. Each number in column I is a square root of the corresponding number in column II.

b. From the table, find the square of 3, 6, 8.

c. From the table, find a square root of 25, 9, 81, 64.

d. Guide pupils to see that while $4 \times 4 = 16$, $(-4) \times (-4)$ also equals 16. Therefore, 16 has two square roots, one positive, one negative: $+4$ and -4.

Have pupils realize that since every perfect square, except zero, is the product of two equal positive numbers *or* of two equal negative numbers, every perfect square has two square roots, a positive square root and a negative square root. (Zero has only one square root: 0.)

e. Tell pupils that the positive square root of a number is called the *principal square root*.

The square roots of 49 are $+7$ and -7. Why? $(+7)(+7) = 49$, and $(-7)(-7) = 49$.

How are the two square roots of 49 related to each other? (They are additive inverses.)

Which is the principal square root of 49? $(+7)$

C. Using the symbol: $\sqrt{\ }$

1. Tell the pupils that the symbol for the principal square root of 49 is $\sqrt{49}$. The symbol $\sqrt{\ }$ is called a radical. The numeral under the radical symbol is called the *radicand*. In the expression $\sqrt{49}$, 49 is the radicand.

2. Since the negative square root of 49 is the additive inverse of the principal square root of 49, tell pupils that the negative square root is represented by a negative sign preceding the radical.

$$\sqrt{49} = 7 \qquad\qquad -\sqrt{49} = -7$$

3. In each of the following, name the radicand and find the indicated square root.

$$\sqrt{36} \qquad -\sqrt{16} \qquad -\sqrt{\tfrac{4}{9}} \qquad \sqrt{1}$$

4. Have pupils consider whether negative numbers have square roots in the set of rational numbers.

   a. Within the set of rational numbers, can you find two equal factors whose product is -9? Since $3 \times 3 = 9$ and $-3 \times -3 = 9$, neither 3 nor -3 is a square root of -9.

   Elicit that in the set of rational numbers we can find no value for $\sqrt{-9}$.

   b. Within the set of rational numbers can you find two factors whose product is -36? -1? $-a^2$?

Have pupils recall that the square of a positive number is positive, and the square of a negative number is positive. Also, that the square of zero is 0. Help them conclude that no negative rational number has a square root in the set of rational numbers.

II. Practice

A. The second power of 6 is _____.

B. The principal square root of 9 is _____.

C. $\sqrt{25}$ is _____ since the second power of 5 is _____.

D. 7^2 = _____ and $\sqrt{?} = 7$.

E. $\sqrt{81}$ is _____, because _____ times _____ is equal to 81.

F. If $10^2 = a^2$, then a = _____ or a = _____.

G. If $36 = b^2$, then $b = \sqrt{?}$ or _____.

H. $\sqrt{\frac{16}{25}}$ = _____.

I. $(-\frac{1}{2})^2 =$

III. Summary

A. Squares of rational numbers are _____ squares.

B. The square root of a number is one of the _____ equal factors of that number.

C. The principal square root is always the _____ square root.

D. The numeral under a radical symbol is called the _____.

E. What new vocabulary did you learn today?

Lesson 81

Topic: Square Root

Aim: To find the principal square root of a perfect square by estimation

Specific Objectives:

To find the principal square root of a perfect square by estimation and trial
To find the length of a side of a square when the area of the square region is given

Challenge: The area of a square region is 1296 square feet. What is the length of each side of the square?

I. Procedure

A. Finding square root by estimation and trial

1. What is the principal square root of 225?

a. Help pupils prepare a table of squares of successive multiples of 10.

n	n × n	n^2
10	10x10	100
20	20x20	400
30	30x30	900
40	40x40	1600
50	50x50	2500
60	60x60	3600
70	70x70	4900
80	80x80	6400
90	90x90	8100

b. To find the principal square root of 225, have pupils estimate:

Since 10^2 is 100 and 20^2 is 400, then $\sqrt{225} > 10$ and $\sqrt{225} < 20$. This can be written as $10 < \sqrt{225} < 20$.

c. Which of the numbers between 10 and 20, when used as a factor twice, will have a product ending in 5? (15)

d. Try 15 × 15.

$$\begin{array}{r} 15 \\ \times 15 \\ \hline 75 \\ 150 \\ \hline 225 \end{array}$$

Then $15^2 = 225$ and $15 = \sqrt{225}$.

2. In a similar way, find the principal square root of

   a. 289 Estimate: $10 < \sqrt{289} < 20$

      Which of the numbers between 10 and 20, when used as a factor twice, will have a product ending in 9?

      Try 13 and 17 ($13 \times 13 = 169$ and $17 \times 17 = 289$).

   b. 324

   c. 625

B. To solve problems involving square roots

1. Refer to the challenge. What is the formula for finding the area of a square? $A = s^2$. What does A represent? What does s represent?

2. Write the formula, replacing A with 1296. $1296 = s^2$.

3. In the formula $A = s^2$, what is an appropriate replacement set for s from all the numbers you know? (the set of positive rational numbers) Since "s" represents the number of units of measure in the side of a square, the value of "s" cannot be negative.

   What replacement for s will make $1296 = s^2$ a true statement? ($\sqrt{1296} = s$)

   Why need we consider only the principal square root?

4. How can we estimate $\sqrt{1296}$?

   $10^2 = 100$ $20^2 = 400$ $30^2 = 900$ $40^2 = 1600$

   Then $\sqrt{1296} > 30$ and $\sqrt{1296} < 40$ or $30 < \sqrt{1296} < 40$.

   Which of the numbers between 30 and 40 when squared would produce a product ending in 6? (34 and 36)

   Try 34
   $$\begin{array}{r} 34 \\ \underline{\times 34} \\ \underline{136} \\ \underline{1020} \\ 1156 \end{array} \quad (1156 \neq 1296)$$

   Try 36
   $$\begin{array}{r} 36 \\ \underline{\times 36} \\ \underline{216} \\ \underline{1080} \\ 1296 \end{array}$$

   Then $1296 = 36^2$ and $36 = s$. The side of the square is 36 ft. long.

II. Practice

A. Find the principal square root of 400, of 900, and of 576.

B. Find the side of a square region whose area is 169 sq. ft.; 1089 sq. ft.

III. Summary

A. Square roots of perfect squares may be found by (estimation) and (trial).

B. To find the side of a square whose area is known, we find the _____ _____ of its area.

Lesson 82

Topic: Rational Numbers: Terminating and Repeating Decimals

Aim: To develop the concept that terminating decimals and non-terminating repeating decimals name rational numbers

Specific Objectives:

To review the meaning of rational number
To express a rational number as a terminating or repeating decimal

Challenge: Write the decimal numeral for $\frac{2}{3}$.

I. Procedure

A. Review meaning of rational number

1. Have pupils recall that a rational number is a number which can be expressed in the form: $\frac{a}{b}$, $b \neq 0$, where a and b represent integers.

2. Express the following rational numbers in the form: $\frac{a}{b}$, $b \neq 0$:

$\frac{3}{4}$, 2, -8, $4\frac{2}{5}$, 1.7 $\left(\frac{3}{4}, \frac{2}{1}, -\frac{8}{1}, \frac{22}{5}, \frac{17}{10}\right)$

B. Expressing a rational number by a terminating or non-terminating repeating decimal

1. Have pupils recall that a rational number may be expressed in decimal notation.

Express $\frac{3}{4}$ in decimal form.

$$\frac{3}{4} = 3 \div 4 \quad \text{or} \quad \begin{array}{r} .75 \\ 4\overline{)3.00} \\ \underline{2\,8} \\ 20 \\ \underline{20} \\ 0 \end{array}$$

$\frac{3}{4} = .75$

Express the following in decimal form: $\frac{5}{8}$, $\frac{22}{5}$.

Tell pupils that such decimals are called terminating decimals because the division ultimately comes to an end when a remainder of zero occurs.

2. Have pupils express $\frac{1}{3}$ in decimal form to 3 places; to 5 places; to 10 places.

   If you continued the division, what digit would be in the 11th place; the 20th place; the 2 millionth?

   Elicit that as far as the division was carried, the next digit in the quotient would also be 3 and the remainder will never be zero. Tell pupils that we can indicate that 3 repeats on and on by placing a bar over the last 3, e.g., $.33\overline{3}$, $.333\overline{3}$, or even $.\overline{3}$.

   In a similar way, have pupils express $\frac{3}{11}$ in decimal form to 4 places; to 6 places; to 8 places.

   Elicit that if the division were continued, the next two digits would be 27. We can indicate that 27 repeats by using a bar, e.g., $.\overline{27}$, $.27\overline{27}$, etc.

3. Refer to the challenge. Elicit that the decimal name for $\frac{2}{3}$, already known by the pupils, is $.66\frac{2}{3}$. This may now be written as $.6\overline{6}$ (or $.\overline{6}$ or $.66\overline{6}$, etc.)

4. Elicit that it seems from the previous examples that a rational number can be named by a terminating decimal or by a non-terminating decimal that repeats. Tell pupils this is true. The teacher may wish, with some classes, to show that the converse of this is true.

5. Do you think the converse of this is true? Does any terminating decimal or non-terminating decimal that repeats name a rational number?

   a. Elicit that the terminating decimals such as .5, .25, .125 can be named by a rational number in the form $\frac{a}{b}$, $b \neq 0$.

      $.5 = \frac{5}{10}$

      $.25 = \frac{25}{100}$

      $.125 = \frac{125}{1000}$

   b. Show pupils that a non-terminating repeating decimal such as $.666\overline{6}$, or $.66\overline{6}$ **also names a rational number.** Let us represent the rational number we seek by n. Then $n = .66\overline{6}$. Since n and $.66\overline{6}$ name the same number, so do $10 \times n$ and $10 \times .66\overline{6}$.

      $$\begin{aligned} 10n &= 6.66\overline{6} \\ n &= .66\overline{6} \end{aligned}$$

      If we subtract n from 10n and $.66\overline{6}$ from $6.66\overline{6}$, we can eliminate $.66\overline{6}$.

To subtract n from 10n we use the distributive property of multiplication over subtraction: $10n - n = (10-1)n$ or $9n$.

Then, by subtraction:

$$\begin{array}{r} 10n = 6.66\overline{6} \\ \underline{n = .66\overline{6}} \\ 9n = 6 \\ n = \frac{6}{9} \text{ or } \frac{2}{3} \end{array}$$

The rational number named by $.66\overline{6}$ is $\frac{2}{3}$.

c. What rational number is named by the non-terminating repeating decimal $.27\overline{27}$?

Let $n = .27\overline{27}$
then $100 \times n = 100 \times .27\overline{27}$

$$\begin{array}{r} 100n = 27.27\overline{27} \\ \underline{n = .27\overline{27}} \\ 99n = 27 \\ n = \frac{27}{99} \text{ or } \frac{3}{11} \end{array}$$

The rational number named by $.27\overline{27}$ is $\frac{3}{11}$.

II. Practice

A. Express each of these rational numbers in the form: $\frac{a}{b}$, $b \neq 0$, where a and b are integers.

13 $\quad 2\frac{6}{7}$ $\quad 12$ $\quad 5\frac{1}{2}$ $\quad 3.6$

B. Express each of these rational numbers by a repeating or terminating decimal.

$\frac{7}{8}$ $\quad 1\frac{1}{3}$ $\quad \frac{4}{7}$ $\quad \frac{4}{5}$

III. Summary

A. A rational number may be expressed by a _____ decimal or by a _____ decimal.

B. What new method did we learn for writing a decimal which repeats.

C. What is the significance of the bar in the following expressions:

$.7\overline{7}$ $\quad .32\overline{32}$ $\quad .\overline{23756}$

Lesson 83

Topic: Irrational Numbers

Aim: To develop the concept of an irrational number

Specific Objectives:

To construct a non-terminating, non-repeating decimal
To learn the characteristics of an irrational number

Challenge: Are there any decimals that do not terminate or repeat?

I. Procedure

A. To construct a non-terminating, non-repeating decimal

1. Have pupils recall that a rational number can be named by a terminating decimal or by a non-terminating repeating decimal.

2. Refer to challenge. Consider the decimal: 0.1234567891011... These digits have been chosen in succession from the set of integers. What will be the next four digits? (12 13) Will this decimal ever terminate? Why not? Will this non-terminating decimal ever repeat? Why not?

3. Answer the challenge. Have pupils make up several non-terminating, non-repeating decimals.

4. Elicit that a non-terminating, non-repeating decimal does not name a rational number. (Rational numbers are named by terminating or non-terminating, repeating decimals.)

B. Characteristics of an irrational number

1. Guide pupils to realize that since the non-terminating, non-repeating decimals do not name rational numbers, a new name is needed for the numbers they do represent.

2. Tell pupils that the numbers named by non-terminating, non-repeating decimals are called irrational numbers. Tell pupils that irrational numbers cannot be expressed in the form $\frac{a}{b}$, where a and b are integers, $b \neq 0$.

3. Have pupils realize that irrational numbers may be positive or negative. For example, .6325447... represents a positive irrational number and -.6325447... represents a negative irrational number.

4. Tell pupils that the union of the set of rational and the set of irrational numbers is called the set of real numbers.

II. Practice

A. Consider the decimal: 0.2468101214...These digits have been chosen in succession from the set of even integers. What will be the next six digits? Explain why this decimal will never terminate. Will this decimal ever repeat? Why? To what set of numbers does this non-repeating, non-terminating decimal belong?

B. Tell whether each of the following is rational or irrational. Explain your answers.

1. $.3232\overline{32}$
2. .823824825826...
3. .75
4. 5

C. To what set of numbers do all the above belong?

III. Summary

A. A non-repeating, non-terminating decimal represents a(n) _____ number.

B. The set of rational numbers and the set of irrational numbers are subsets of the set of _____ numbers.

C. What new vocabulary did we learn today?

(irrational, real numbers)

Lesson 84

Topic: Square Root

Aim: To approximate the square root of a non-perfect square

Specific Objectives:

To learn that the square root of a non-perfect square is an irrational number
To find a rational approximation of the square root of a non-perfect square

Challenge: What is the principal square root of 18?

I. Procedure

A. The square root of a non-perfect square is an irrational number

1. Refer to challenge. Have pupils recall that when a number can be factored into exactly two equal rational factors, each of the equal factors is a square root of the number.

2. Is there any integer which when used as a factor twice will result in a product of 18?

   since $\sqrt{16} = 4$ then $\sqrt{18} > 4$
   since $\sqrt{25} = 5$ then $\sqrt{18} < 5$
   $\therefore 4 < \sqrt{18} < 5$

   Have pupils conclude that there is no integer which when used as a factor twice will give us 18 as a product.

3. Is there any rational number between 4 and 5 which used as a factor twice will result in 18 as a product?

   Have pupils try various rational numbers such as 4.2, 4.36, 4.123, etc., and list results on the board.

   $(4.2)^2 = 17.64$

   $(4.36)^2 = ?$

   $(4.123)^2 = ?$, etc.

   Pupils will discover that none of the numbers they tried will give a product of 18.

4. Have pupils recall the meaning of a perfect square. (the square of a rational number)

   From our work, it appears there is no rational number whose square is 18. If this is true, is 18 a perfect square? Why not? (Because a perfect square is the result of squaring a rational number.)

5. Tell the pupils that it will be proved in later courses that numbers such as 18 are not perfect squares and therefore do not have rational square roots. Such numbers can be called non-perfect squares. The square root of a non-perfect square is an irrational number.

B. Find a rational approximation of the square root of a non-perfect square

1. Since the square root of a non-perfect square is an irrational number, we will try to find a rational approximation of $\sqrt{18}$. By "a rational approximation" we mean a rational number which will approximate the value of an irrational number.

2. Refer to A-2, $4 < \sqrt{18} < 5$ means that the rational approximation of $\sqrt{18}$ will be 4 and some decimal fraction. What does $(4.1)^2$ equal? (16.81) What does $(4.2)^2$ equal? (17.64) What does $(4.3)^2$ equal? (18.49)

   Between which of these rational approximations will the square root of 18 lie? ($4.2 < \sqrt{18} < 4.3$)

3. From your multiplications, $(4.2)^2 = 17.64$ and $(4.3)^2 = 18.49$.

   What is the difference between 18 and 17.64?
   What is the difference between 18.49 and 18?
   Which rational approximation would you say is the closer to $\sqrt{18}$? (4.2)
   Then we will say that $\sqrt{18} = 4.2$ to the nearest tenth.

   Note to Teacher: The pupils should realize that the rational approximation to the square root of a non-perfect square can be found to hundredths or thousandths, etc. For example, the rational approximation of $\sqrt{18}$ to 15 decimal places is 4.242640687119285. In any problem calling for the square root of a non-perfect square, the number of decimal places required will always be indicated. In this material we shall not go beyond tenths place.

II. Practice

A. Between what two consecutive integers does each of the following lie? $\sqrt{7}$ (between 2 and 3), $\sqrt{72}$, $\sqrt{28}$

B. Find the value of $(5.2)^2$, $(7.8)^2$, $(3.4)^2$.

C. Find to the nearest tenth: $\sqrt{17}$, $\sqrt{5}$, $\sqrt{54}$, $\sqrt{76}$.

III. Summary

A. Why is $\sqrt{2}$ an irrational number?

B. Explain what is meant by a rational approximation of an irrational number.

Lesson 85

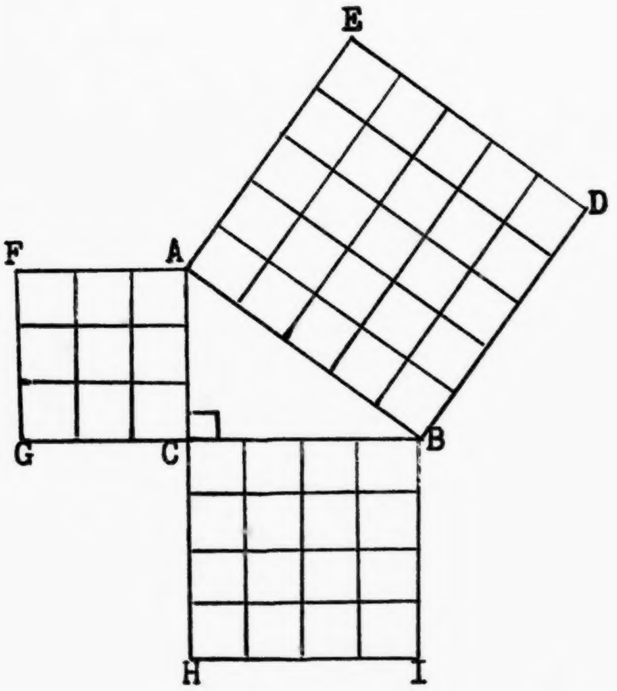
Topic: The Pythagorean Relationship

Aim: To discover the Pythagorean relationship in a right triangle

Specific Objectives:

To review the meaning of: right triangle, hypotenuse, legs
To discover the Pythagorean relationship among the sides of any right triangle

Note: It is suggested that the teacher prepare and distribute rexographed sheets containing the following diagrams:



Challenge: What relationship can you discover among the area measures of squares ABDE, ACGF, AND CBIH?

I. Procedure

A. Review meaning of right triangle, hypotenuse, legs

1. In the figure shown in the challenge, what kind of polygon is ACB?

2. In right triangle ACB, which is the largest angle?

3. Which is the longest side of triangle ACB? (hypotenuse)

4. If angle C is a right angle, name the hypotenuse. Name each leg.

5. What is the unit measure of side AB? side CB? side AC? (AB = 5 units, CB = 4 units, AC = 3 units)

B. To discover the Pythagorean relationship

1. How many square units are there in each of the regions ABDE, ACGF, CBIH?

2. What relationship can you discover among these area measures? (25 = 9 + 16)

3. Have pupils realize that the area of each square region equals the square of the measure of one of its sides, i.e., $9 = 3^2$; $16 = 4^2$; $25 = 5^2$ and therefore 25 = 9 + 16 could be expressed as $5^2 = 3^2 + 4^2$.

4. In a manner similar to the challenge diagram, have each pupil draw on squared paper a right triangle with legs 6 units and 8 units in length.

   Draw squares on the legs of the triangle and find the area of each square region. Use another piece of squared paper to measure the hypotenuse. Does $10^2 = 6^2 + 8^2$? This example illustrates the Pythagorean relationship named for the Greek mathematician, Pythagoras. It may be stated: In a right triangle, the square of the measure of the hypotenuse equals the sum of the squares of the measures of the legs.

5. Using a and b to represent the lengths of the legs of a right triangle, and c to represent the length of the hypotenuse, how can we express the area of the square region drawn on side a? (a^2); on side b? (b^2); on side c? (c^2)

   How can we express the Pythagorean relationship in terms of a, b, and c? ($c^2 = a^2 + b^2$)

   Note to Teacher: Have pupils realize that $c^2 = a^2 + b^2$ is a generalization of the Pythagorean relationship.

II. Practice

A. Have pupils draw right triangles in different positions and have them name the hypotenuse and legs of each triangle.

B. Listed below are the lengths of the sides of several right triangles. Have pupils check to see that the relationship $c^2 = a^2 + b^2$ holds.

1. 5, 12, 13 Does $13^2 = 5^2 + 12^2$?

2. 8, 15, 17

3. 7, 24, 25

4. 12, 16, 20

III. Summary

A. In a right triangle, what name is given to the side opposite the right angle? to the sides opposite the acute angles?

B. In a right triangle, which is the longest side?

C. What relationship seems to exist among the sides of a right triangle? Why is it called the Pythagorean relationship?

D. Using the letters a and b to represent the measures of the legs, and c to represent the measure of the hypotenuse of a right triangle, how can we express the Pythagorean relationship?

Lesson 86

Topic: Pythagorean Relationship

Aim: To use the Pythagorean relationship

Specific Objectives:

To review finding the square root of a number
To use the Pythagorean relationship to find the hypotenuse of a right triangle

Challenge: What is the hypotenuse of a right triangle whose legs measure 8" and 15"?

I. Procedure

A. Review of finding square root

1. The square roots of 25 are +5 and -5. Why?

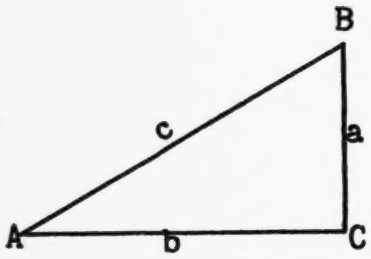
2. Which is the principal square root of 25?

3. Find the principal square root of 196.

4. Find $\sqrt{72}$ to the nearest tenth.

B. Using the Pythagorean relationship to find the hypotenuse

Note to Teacher: In order to facilitate problem solving, it is suggested that the pupils find square roots or square root approximations by using the tables in their textbooks. The work done by the pupils in estimating and approximating square roots will reinforce their understanding and help them to avoid errors in reading the tables. It will also help the pupils to find answers when the tables are not available.



1. Refer to the challenge.
Draw a figure representing the right triangle.

2. Elicit that $\overline{AC}$ and $\overline{BC}$ are the legs of the right triangle ACB and that $\overline{AB}$ is the hypotenuse.

3. Have pupils recall the generalization of the Pythagorean relationship, $c^2=a^2+b^2$ and make the appropriate numeral replacements: $c^2=8^2+15^2$.

4. Solving the above equation: $c^2 = 64 + 225$
$c^2 = 289$

What is the value of c? (17 or -17)
Which value of c do we consider for our answer? (c = 17) Why?
Therefore, the length of the hypotenuse is 17".

5. If the lengths of the legs were 2' and 3', how long would the hypotenuse be?

   $c^2 = 3^2 + 2^2$
   $c^2 = 9 + 4$
   $c^2 = 13$

6. If $c^2 = 13$, what kind of number will be the value of c? (irrational) Why? (Because no rational number used as a factor twice will result in 13.)

7. Find the square root of 13 to the nearest tenth.

   From the table in the textbook, the three-place decimal rational approximation for $\sqrt{13}$ is 3.606. $\sqrt{13}$ to the nearest tenth is 3.6.

II. Practice

A. What is the principal square root of 400?

B. Find $\sqrt{74}$ to the nearest tenth

   1. by approximation

   2. by using the table in the textbook.

C. Find the length of the hypotenuse of a right triangle when the measures of the legs are:

a = 15 inches	b = 20 inches
a = 9 centimeters	b = 40 centimeters
a = 1 meter	b = 1 meter

III. Summary

A. What is meant by the principal square root of a number?

B. How would you express in words the Pythagorean relationship?

Lesson 88

Topic: Irrational Numbers

Aim: To show that a point on the number line may be associated with an irrational number

Specific Objectives:

To find, by approximation, a point on a number line associated with an irrational number

To understand that every point on the number line has either a rational or an irrational number associated with it

Challenge: What point on the number line would you associate with $\sqrt{16}$? $\sqrt{25}$? $\sqrt{18}$?

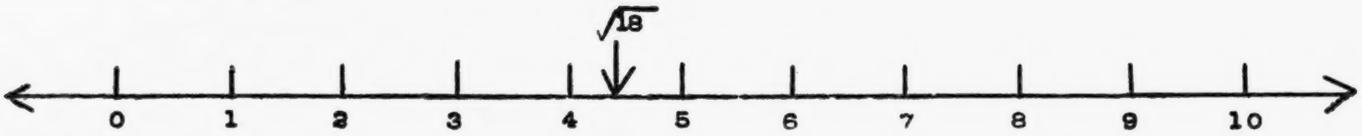
I. Procedure

A. Associating a point on a number line with an irrational number

1. Have pupils recall that they have associated points on a number line with whole numbers, with integers, and with rational numbers.

2. Refer to challenge. Elicit that since $\sqrt{16} = 4$ and $\sqrt{25} = 5$, $\sqrt{18}$ is greater than 4 and less than 5; that is, $4 < \sqrt{18} < 5$.

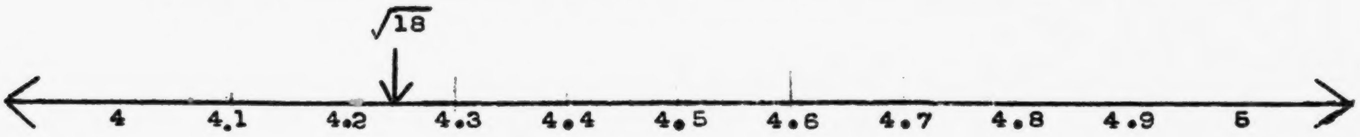
   On the number line, the point associated with $\sqrt{18}$ is between the points associated with 4 and 5.



3. Have pupils recall from a previous lesson that in approximating $\sqrt{18}$ to one decimal place

   $(4.2)^2 = 17.64$
   $(4.3)^2 = 18.49$
   $\therefore 4.2 < \sqrt{18} < 4.3$

   Imagine a magnifying glass used to enlarge the interval on the number line from 4 to 5. Divide this interval into ten equal parts. The point associated with $\sqrt{18}$ is between 4.2 and 4.3.



4. In approximating $\sqrt{18}$ to two decimal places, we find that

   $(4.24)^2 = 17.9776$
   $(4.25)^2 = 18.0625$
   $\therefore 4.24 < \sqrt{18} < 4.25$

c. Using the Pythagorean relationship

$$\begin{aligned} c^2 &= a^2 + b^2 \\ (12)^2 &= (11)^2 + b^2 \\ 144 &= 121 + b^2 \\ 23 &= b^2 \\ 4.796 &= b \end{aligned}$$

d. Therefore, the distance to the nearest foot is 5 feet.

II. Practice

A. Using the Pythagorean relationship, find the length of the third side to the nearest integer.

a = 5, c = 13

b = 11, c = 14

B. A pole 40 feet in height is steadied by a guy wire 41 feet in length attached at the top of the pole. How far from the foot of the pole is the foot of the guy wire?

C. A rectangular sheet of art paper is 24" long and 7" wide. What is the length of the longest straight line that can be drawn on this paper?

III. Summary

A. Explain how you would use the Pythagorean relationship to find the length of one leg of a right triangle if the lengths of the hypotenuse and the other leg are known.

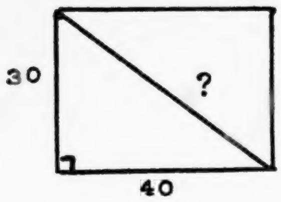
B. List the steps you would take to solve a verbal problem in which the Pythagorean relationship is used.

c. Have pupils use the table to find the square root of 51 (7.141).

d. Elicit that 7.141 = 7.1 to the nearest tenth. Therefore, the length of the leg is 7 inches.

B. Using the Pythagorean relationship to solve verbal problems

1. A rectangular lot is 30 ft. wide and 40 ft. long. If a boy cuts diagonally across the lot, how many feet would he save over the distance covered by walking along the sidewalk?



a. Have pupils draw a diagram representing the conditions in the problem.

b. Elicit that to answer the question posed by the problem, it will be necessary to find the length of the diagonal path.

c. Have pupils realize that since the diagonal path cuts the rectangular lot into two right triangles, the Pythagorean relationship may be used to find its length.

d. $c^2 = a^2 + b^2$
$c^2 = (40)^2 + (30)^2$
$c^2 = 1600 + 900$
$c = 50$ Length of the diagonal path is 50 ft.

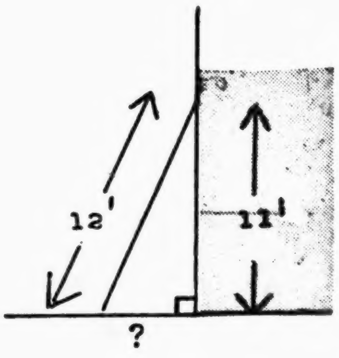
e. Elicit that if the boy had walked along the sidewalk, he would have walked 70 ft.

f. Answer to problem: 70 - 50 = 20 ft. saved.

2. A 12 ft. ladder is placed against a building so that it just reaches a window 11 ft. from the ground. How far is the bottom of the ladder from the base of the building? Round off your answer to the nearest foot.

a. Have pupils draw a diagram representing the conditions in the problem.

b. Elicit that the ladder forms a right triangle with the side of the building and the ground.



Lesson 87

Topic: Pythagorean Relationship

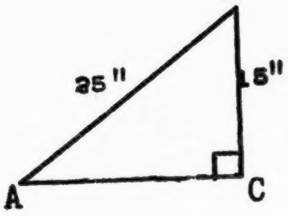
Aim: To use the Pythagorean relationship

Specific Objectives:

To use the Pythagorean relationship to find the length of one leg of a right triangle

To use the Pythagorean relationship to solve "verbal problems" involving right triangles

Challenge: In the right triangle ABC shown at the right, what is the length of $\overline{AC}$?



I. Procedure

A. To find the length of one leg when the lengths of the hypotenuse and the other leg are known

1. Refer to challenge. Elicit that since triangle ABC is a right triangle, the Pythagorean relationship among the sides will hold.

2. Have pupils recall the generalization of the Pythagorean relationship $c^2 = a^2 + b^2$ and make the appropriate replacements.

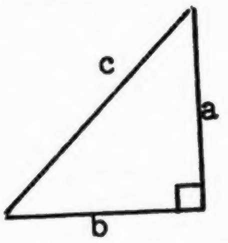
$$\begin{aligned} c^2 &= a^2 + b^2 \\ (25)^2 &= (15)^2 + b^2 \\ 625 &= 225 + b^2 \end{aligned}$$

3. What property of equation solving could we use to find the value of b^2?

4. If $\underline{400} = b^2$, what is the value of b? Therefore, the length of $\overline{AC}$ is 20 inches.

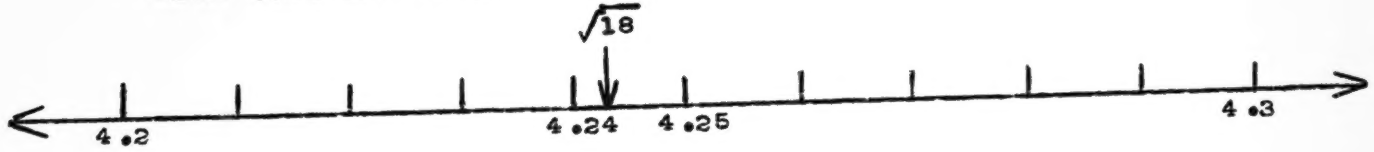
5. If the hypotenuse of a right triangle is 10 inches, and one leg is 7 inches, find the length of the other leg. Give the answer to the nearest inch.

   a. Have pupils draw a diagram to represent the conditions in the problem.



   b.
$$\begin{aligned} c^2 &= a^2 + b^2 \\ (10)^2 &= (7)^2 + b^2 \\ 100 &= 49 + b^2 \\ 51 &= b^2 \end{aligned}$$

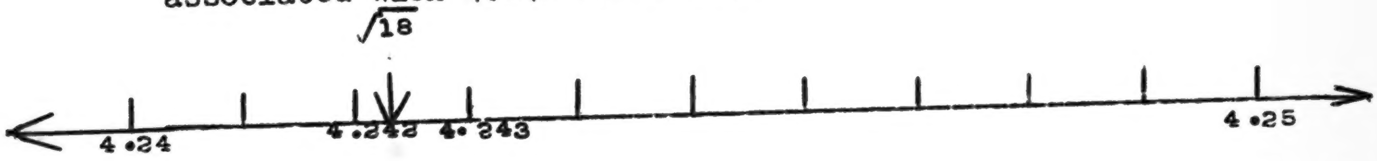
Imagine a magnifying glass enlarging the interval on the number line from 4.2 to 4.3. Divide this interval into ten equal parts. The point associated with $\sqrt{18}$ is between the points associated with 4.24 and 4.25.



5. In the rational approximation of $\sqrt{18}$ to three decimal places, we find that

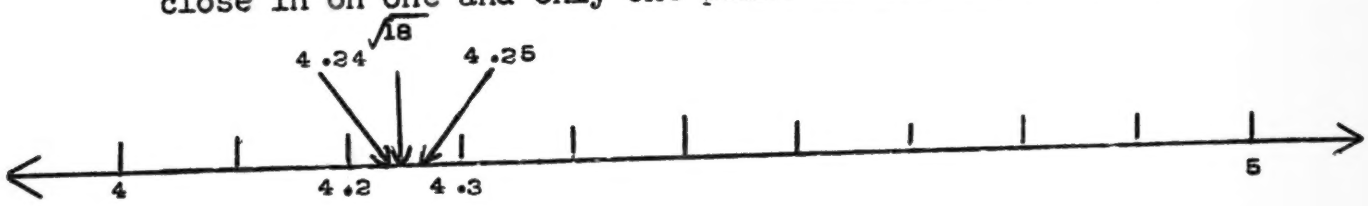
$$(4.242)^2 = 17.994564$$
$$(4.243)^2 = 18.003043$$
$$\therefore 4.242 < \sqrt{18} < 4.243$$

Imagine a magnifying glass enlarging the interval on the number line from 4.24 to 4.25. Divide this interval into ten equal parts. The point associated with $\sqrt{18}$ is between the points associated with 4.242 and 4.243.



Have pupils realize that in this process each magnified interval lies within the preceding interval.

6. Have pupils note that as the number of decimal places in the rational approximation increases, the interval on the number line between the points associated with each successive pair of numbers becomes smaller and smaller. As the rational approximations close in on the irrational number represented by $\sqrt{18}$, the points associated with these approximations get closer and closer. It seems then that they will eventually close in on one and only one point on the number line.



7. Tell pupils that in a similar manner, it can be shown that there is a unique point on the number line associated with each irrational number.

B. The real number line

1. Elicit that the set of whole numbers and the set of integers are subsets of the set of rational numbers.

2. Have pupils recall that the set of real numbers consists of the union of the set of rational numbers and the set of irrational numbers.

3. Tell pupils that every point on the number line is associated with a real number (rational or irrational) and each real number (rational or irrational) is associated with one and only one point on the number line.

   Therefore, when we represent the set of real numbers on the number line, the points associated with these real numbers have no spaces between them.

   Only with the set of real numbers do we have a one-to-one correspondence between the set of numbers and the set of all points on the line. Such a number line is called the real number line.

II. Practice

   A. Consider the placement on the number line of the irrational number 1.732...

      1. Between what two integers will it lie? (1 and 2)

      2. Between what two one-place decimal rational approximations will it lie? (1.7 and 1.8)

      3. Between what two two-place decimal rational approximations will it lie? (1.73 and 1.74)

   B. The $\sqrt{2} = 1.41421...$ Using a three-place decimal rational approximation indicate on a number line a point which approximates $\sqrt{2}$.

III. Summary

   A. What are some subsets of the set of real numbers?

   B. Describe the process by which we locate a point on the number line associated with an irrational number.

CHAPTER IX

GRAPHS

The procedures suggested in this chapter will help pupils to develop an understanding of graphing an open sentence on the number plane.

Among the important concepts developed in the chapter are:

meaning of the symbols $\geq$ and $\leq$
graphing on the real number line inequalities in one variable
meaning of an ordered number pair; locating a position in a plane by means of an ordered number pair
finding a point in a number plane which corresponds to an ordered number pair
graphing the solution set of an equation in two variables
graphing a formula in two variables

The beginning lessons introduce the graphing of the inequalities: $x \geq n$, $x \leq n$ on the real number line. A comparison is made between graphs of such inequalities when the replacement set is the set of integers and when it is the set of real numbers.

The solution of open sentences in one variable is briefly reviewed and the solution of an open sentence in two variables is introduced by testing various numbers of the replacement set for the variables. The pupil is then guided to discover the meaning of an ordered number pair.

The pupil learns first to locate a position in a plane, as, for example, a position in a classroom by means of an ordered number pair. This knowledge is then applied to situations, such as locating points of interest on a city map.

The pupil then learns to associate a point in the number plane with an ordered number pair and to graph points on a number plane. At this stage, a systematic process for finding solutions of an equation in two variables is presented. The solution set is then graphed on the number plane.

The importance of the replacement set for the variable is developed by showing the effect of a change in the replacement set on the graph of an equation. Pupils are led to understand the need for choosing an appropriate replacement set in graphing such formulas as $p = 4s$, the perimeter of a square and $p = 3s$, the perimeter of an equilateral triangle.

CHAPTER IX

GRAPHS

Lessons 89-98

Lesson 89

Topic: The Inequality

Aim: To solve, by testing the members of the replacement set, inequalities of the form $x \geq a$, $x \leq a$

Specific Objectives:

To understand the meaning of the symbols: $\leq$, $\geq$
To solve inequalities of the form $x \geq a$, $x \leq a$

Challenge: To vote in New York State, a citizen must be 21 years of age or older. How would you represent this condition using algebraic symbols?

I. Procedure

A. To understand the meaning of the symbols: $\geq$, $\leq$

1. Refer to challenge. If x represents the number of years in the age of the citizen then, either

   $x = 21$ or
   $x > 21$

   Why are the two sentences needed?

2. How could we combine the two sentences above into one sentence? ($x = 21$ or $x > 21$) Show pupils that the sentence: " $x = 21$ or $x > 21$" is conventionally written: $x \geq 21$ and is read: x is greater than or equal to 21.

3. Since the symbol $\geq$ means "greater than or equal to," how can we represent "less than or equal to"?

   If the automobile speed limit on a parkway is 50 mph, and if y represents the speed of the car, express all the legal rates of speed in mathematical symbols.

   Since $y = 50$ is legal
   and $y < 50$ is legal

   Elicit that $y \leq 50$ is a representation of a speed which must be less than or equal to 50 mph.

4. Express each of the following as a mathematical sentence:

   r is greater than or equal to 10
   t is less than or equal to 2

B. Solving inequalities of the form $x \geq a$, $x \leq a$

1. Name four elements from the set of positive integers which make the inequality $x \geq 4$ a true statement.

   a. How may the replacement set for x be represented? $\{1,2,3,\ldots\}$

   b. Name four replacements that will make the sentence $x \geq 4$ true.

   c. Explain why 4 and 5, when used as replacements, each makes the sentence true.

   d. Name one replacement that will make the sentence false.

   e. Elicit that the solution set is sometimes called the truth set.

2. Consider $x \leq 10$ where the replacement set for x is the set of integers.

   a. How may the replacement set for x be represented? $\{\ldots, -2,-1,0,1,2,\ldots\}$

   b. Name four replacements that will make the sentence $x \leq 10$ true.

   c. Explain why both 10 and -10, when used as replacements, will make the sentence true.

II. Practice

A. Using the replacement set $\{-5,-4,-3,\ldots,5\}$, find the solution set of each of the following:

$x \geq 2$ ($\{2,3,4,5\}$)
$x \leq 0$ ($\{-5,-4,-3,-2,-1,0\}$)
$x \geq 5$ ($\{5\}$)
$x \leq -7$ ($\{\ \}$ or $\emptyset$)

B. Suppose n represents any number in the set of positive even integers. Name two replacements for n that will make $2n + 3 \geq 11$.

C. What is the solution set of the open sentence in B? ($\{4,6,8,\ldots\}$)

III. Summary

A. What does the symbol $\geq$ mean?
B. What does the symbol $\leq$ mean?
C. State a life situation using the expression "greater than or equal to." Represent the situation by means of the symbol $\geq$.

Lesson 90

Topic: The Inequality

Aim: To graph the solution set of an inequality in one variable on the number line

Specific Objectives:

To review graphing the solution set of an inequality in one variable
Replacement set: set of integers

To graph the solution set of an inequality in one variable
Replacement set: set of real numbers

Challenge: What is the difference in the graph of the solution set of $x > 3$ when the replacement set is the set of integers and when the replacement set is the set of real numbers?

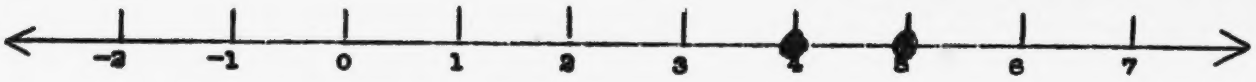
I. Procedure

A. Review the graph of the solution set of $x > 3$, replacement set: the set of integers

1. Have pupils recall that the solution set of $x > 3$ can be shown by means of a graph. For example,

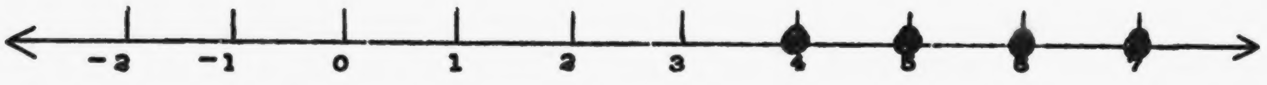
a. If the replacement set is $\{1,2,3,4,5\}$, what is the solution set of $x > 3$? ($\{4,5\}$)

b. Graph the solution set on the number line by indicating the members of the solution set with heavy dots.



2. a. If the replacement set is the set of integers, what is the solution set of $x > 3$? ($\{4,5,6,\ldots\}$)

b. Graph the solution set.



Have pupils note the difference between the two graphs. Lead pupils to realize that since the last point named on the line is included on the graph, this shows that the positive integers greater than 7 are also in the solution set.

3. Using a similar procedure, have pupils graph $x < 3$, if

   a. the replacement set is $\{-5, -4, -3, -2, -1, 0, 1\}$;

   b. the replacement set is $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

B. To graph the solution set of $x > 3$
Replacement set: set of real numbers

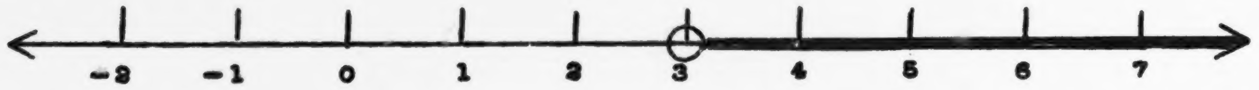
1. When the replacement set is the set of real numbers, what is the solution set of $x > 3$? (the set of all real numbers greater than 3)

   Tell pupils we can indicate this solution set by using set-builder notation. We write $\{x | x > 3, x \in R\}$ and read: the set of all x such that x is greater than 3 and x belongs to the set of real numbers. Notice the vertical bar is read "such that."

2. Have pupils recall that the union of the set of rational numbers and the set of irrational numbers form the set of real numbers. Recall that there is a one-to-one correspondence between the set of real numbers and the points on the number line.

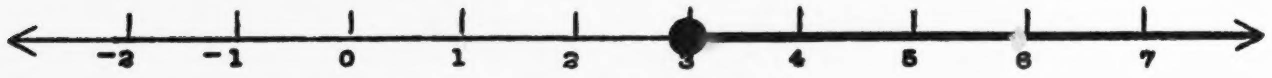
3. Since every real number greater than 3 is a member of the solution set, we indicate their graphs by darkening part of the number line.

   To show that 3 is not included in the solution set, a circle is drawn about the point corresponding to 3.



4. Answer the challenge.

5. Use as the replacement set the set of real numbers. What is the solution set for $x \geq 3$? (3 and all real numbers greater than 3) Tell pupils that since 3 is included in the solution set, we place a shaded circle on the point corresponding to 3 and a heavy line to represent all other members of the solution set.



6. Using a similar procedure, have pupils graph $\{x | x \leq 3, x \in R\}$.

II. Practice

A. Graph the solution sets of each of the following inequalities.
Replacement set: set of real numbers

1. $x > 5$
2. $x < -1$
3. $4x > -8$
4. $3x < 9$

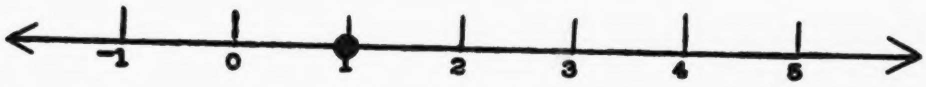
B. Graph the solution sets of each of the following inequalities using the set of real numbers as the replacement set.

1. $x \leq 6$

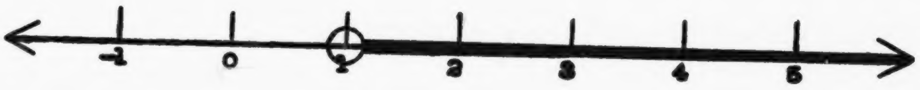
2. $2x \geq -2$

C. Consider the following graphs:
Replacement set: set of real numbers

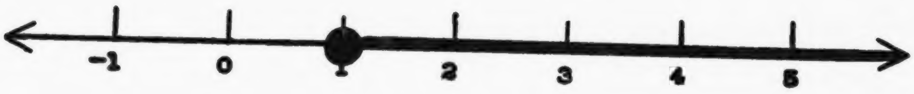
1.



2.



3.



Write an open sentence for which each of the above is a graph of the solution set.

III. Summary

A. Explain: If I is the set of integers, the graph of $\{x | x \leq 3, x \in I\}$ is a set of distinct points on the number line.

B. Explain: If R is the set of real numbers, the graph of $\{x | x \leq 3, x \in R\}$ is a ray whose end point is the point associated with the number 3 on the number line.

C. What is the significance of the unshaded circle when used as part of the graph of the solution set of an inequality on the real number line?

D. Does the graph of $\{x | x \leq 6\}$ include the point associated with the number 6? Explain your answer.

Lesson 91

Topic: Open Sentences in Two Variables

Aim: To develop the meaning of an ordered pair of numbers

Specific Objectives:

To review the solution sets of open sentences in one variable
To introduce the solution sets of open sentences in two variables
To develop the meaning of an ordered pair of numbers

Challenge: Explain why $\{4,3\} = \{3,4\}$ but $(4,3) \neq (3,4)$.

I. Procedure

A. Solution sets of open sentences in one variable

1. Have pupils recall that an open sentence is one that cannot be judged true or false.

2. Elicit that the truth or falsity of an open sentence such as $2n = 14$ cannot be determined until n is replaced by the name of a number from the replacement set.

3. If the replacement set is the set of rational numbers, what is the solution set of:

$n + 3 = 7$? $2n = 14$? $3n = -10$?

B. Solution of open sentences in two variables

Note to Teacher: Tell pupils that from now on unless stated otherwise, the replacement set will be the set of real numbers.

1. Have pupils consider the open sentence: $x + 2y = 10$.

2. Elicit that such an open sentence requires replacements for both x and y.

3. Have pupils try various replacements for x and y that will make a true statement of the open sentence $x + 2y = 10$.

x	y	x + 2y = 10	
10	0	$10 + 2(0) = 10$	True
2	5	$2 + 2(5) = 10$	False
8	1	$8 + 2(1) = 10$	True
3	4	$3 + 2(4) = 10$	False
4	3	$4 + 2(3) = 10$	True

4. Show pupils that by choosing a replacement for one variable, we can find the replacement for the other variable which will make the statement true in a systematic way.

   If we choose 6 as a replacement for x, we have

$$\begin{aligned} x + 2y &= 10 \\ 6 + 2y &= 10 \\ 2y &= 4 \\ y &= 2 \end{aligned}$$

   Have pupils check

$$\begin{aligned} x + 2y &= 10 \\ 6 + 2(2) &= 10 \\ 6 + 4 &= 10 \\ 10 &= 10 \end{aligned}$$

   If we choose 7 as a replacement for x, we have

$$\begin{aligned} x + 2y &= 10 \\ 7 + 2y &= 10 \\ 2y &= 3 \\ y &= \tfrac{3}{2} \text{ or } 1\tfrac{1}{2} \end{aligned}$$

   Have pupils check

$$\begin{aligned} x + 2y &= 10 \\ 7 + 2(\tfrac{3}{2}) &= 10 \\ 7 + 3 &= 10 \\ 10 &= 10 \end{aligned}$$

   Show pupils how to arrange their work.

x	x + 2y = 10	y
6	6 + 2y = 10	2
7	7 + 2y = 10	$\frac{3}{2}$

5. Lead pupils to realize that although a solution of an open sentence in one variable is a number, a solution of an open sentence in two variables is a pair of numbers.

C. Ordered number pairs

1. Tell pupils that number pairs such as 10 (the replacement for x) and 0 (the replacement for y) are conventionally written (10,0) with the value for x given first.

2. Refer to the table in B-4. Have pupils note that when x is replaced by 4 and y by 3, the resulting statement is true,

but when x is replaced by 3 and y by 4, the resulting statement is false. Have pupils conclude that $(4,3) \neq (3,4)$ and that, therefore, the order is important.

3. Return to challenge. $\{4,3\} = \{3,4\}$ because if the members of the set are the same, the sets are equal regardless of order. $(4,3) \neq (3,4)$ because $(4,3)$ means $x = 4$, $y = 3$ and $(3,4)$ means that $x = 3$, $y = 4$ and order is important.

4. Tell pupils that a pair of numbers in which the order is important is called an ordered pair of numbers.

II. Practice

A. Tell whether the given ordered pair of numbers is a solution of the open sentence.

$x + y = 3$	$(2,1)$	Yes, because $2+1 = 3$
$a + 4b = 9$	$(3,2)$	No, because $3+4(2) \neq 9$
$3x + y = 6$	$(1,3)$	
$2x - 2y = 10$	$(2,-3)$	

B. Find several ordered pairs of numbers which will make the following true:

$\frac{1}{2}x + y = 12$
$x - y = 16$
$x + 2y = -4$

III. Summary

A. What vocabulary have you learned today?

B. What is the difference between a solution of an open sentence in one variable and a solution of an open sentence in two variables?

C. Why is a solution of an open sentence in two variables called an ordered pair of numbers?

Lesson 92

Topic: Ordered Pairs of Numbers

Aim: To locate a position or point by means of an ordered pair of numbers

Specific Objectives:

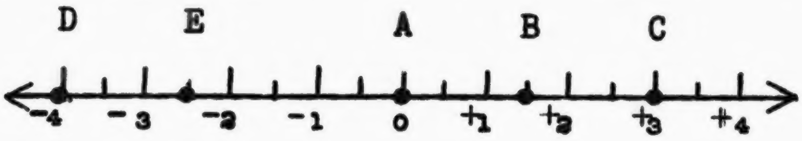
To review locating a point on a number line
To learn to locate a position or point by means of an ordered pair of numbers

Challenge: In your classroom, who is sitting in seat (5,2)? in seat (2,5)?

I. Procedure

A. Locating a point on a number line

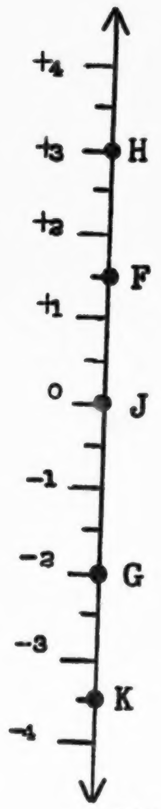
1. Have pupils consider the number line below:



What real number is associated with each point A,B,C,D,E?

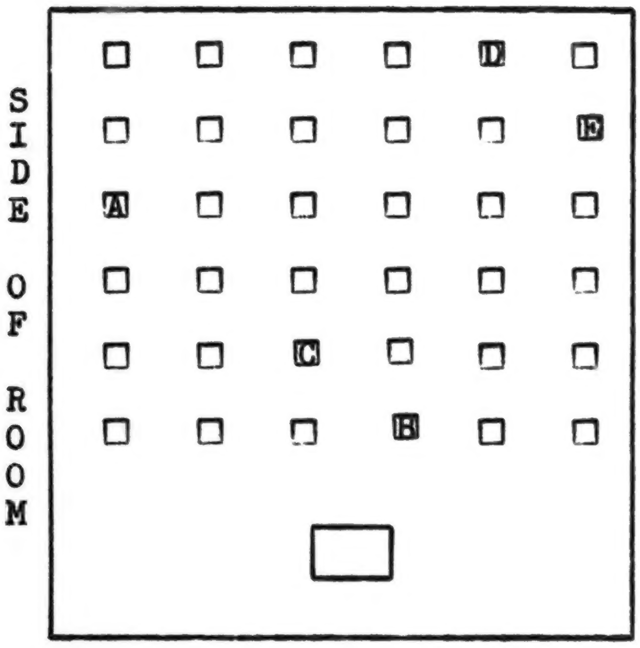
Tell the pupils that the number which corresponds to a point on the number line is called the coordinate of the point.

2. What is the coordinate of each of the points H,J,K,G,F on this vertical number line.



B. Associating an ordered number pair with a position or point in a real life situation

1. Have pupils examine a plan of an imaginary classroom such as the one below.



Front of Room

Have pupils suggest the directions that must be given to locate the following pupils in the classroom.

A. Albert	D. David
B. Betty	E. Edward
C. Carla	

2. Elicit that a pair of numbers is necessary to locate a pupil's position because both the row and the seat must be identified.

   We will agree that the number of the row from the side of the room will be given first, and the number of the seat from the front of the room will be given second.

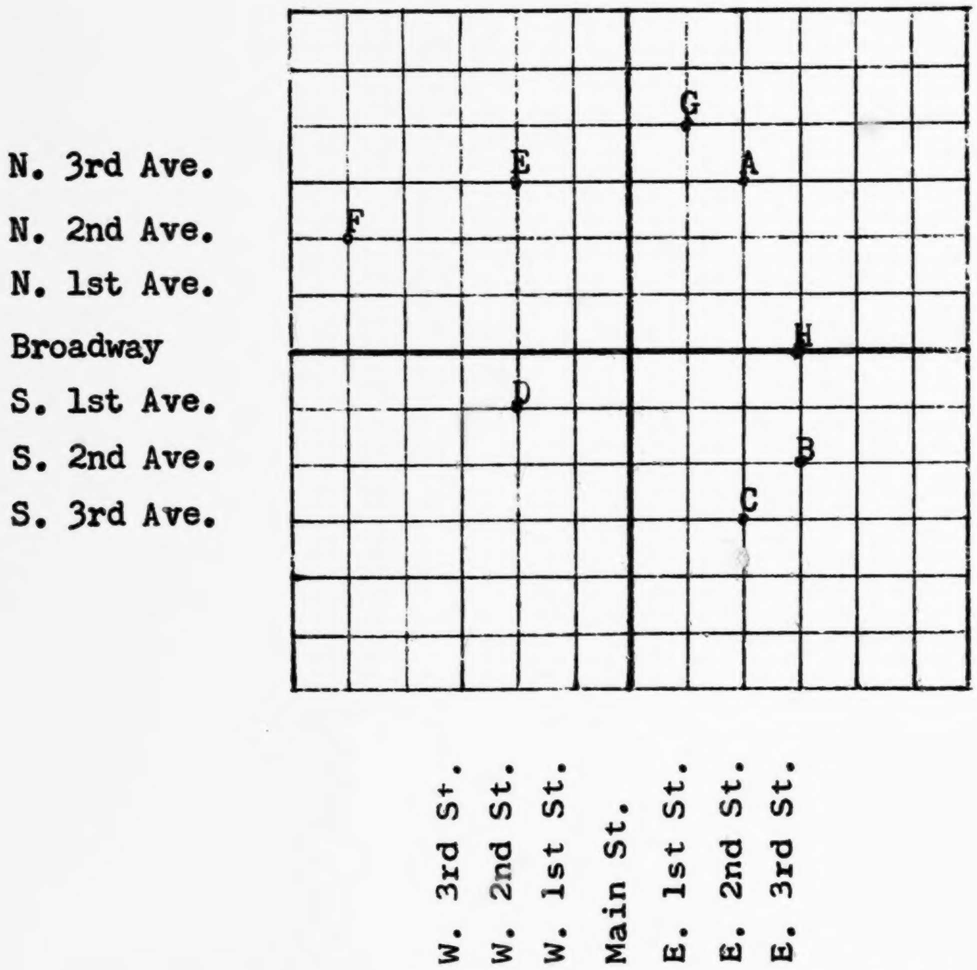
   The pupil located by the first row, fourth seat (Albert) is not the same as the pupil located by the fourth row, first seat (Betty). Therefore, order is important and we need an ordered number pair.

3. If we represent the location of pupil A as (1,4), how would you represent the location of pupil B? pupil C? pupil D? pupil E?

   Using the symbolism for the ordered pair of numbers, have pupils locate their positions in the classroom.

4. Answer the challenge.

II. Practice


N. 3rd Ave.
N. 2nd Ave.
N. 1st Ave.
Broadway
S. 1st Ave.
S. 2nd Ave.
S. 3rd Ave.
G
E
A
F
H
D
B
C
W. 3rd St.
W. 2nd St.
W. 1st St.
Main St.
E. 1st St.
E. 2nd St.
E. 3rd St.

A. Using the map above, have pupils suggest the directions that must be given to find the following places in the town. Use the intersection of Main Street and Broadway as a reference or starting point.

A. Art Museum (2 blocks E, 3 blocks N)
B. Bayview School
C. City Hall
D. Dan's Market
E. Eastern Air Lines Office
F. Fairmont Hospital
G. General Post Office
H. Hall of Records

B. Locate the points G and A by means of ordered pairs of numbers giving the street first and avenue second. Answer: G (E1, N4)

III. Summary

A. How many directions do you need to locate a point on a number line?
B. How many directions do you need to locate a seat in a classroom?

Lessons 93 and 94

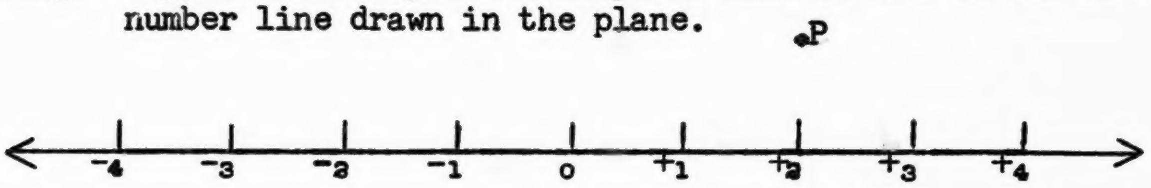
Topic: The Graph of a Point in a Number Plane

Aim: To learn that points in a plane and ordered pairs of numbers may be made to correspond

Specific Objectives:

To associate a point in the number plane with an ordered pair of numbers
To match a pair of numbers with a point in a plane

Challenge: Suppose P is a point in a plane that is not on the real number line drawn in the plane.



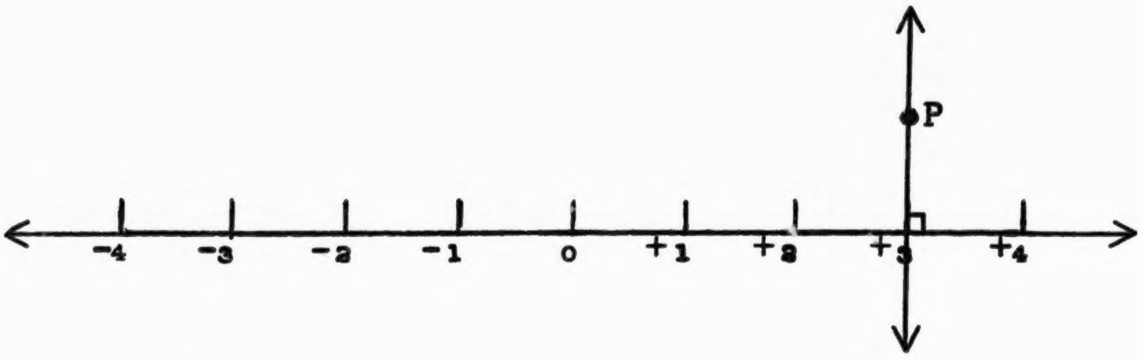
How would you locate the position of point P?

I. Procedure

A. To associate a point in the number plane with an ordered pair of numbers

1. In reference to the challenge, elicit that P is directly above some point on the number line.

2. How can we find the point on the number line directly below point P? (Draw a line through P perpendicular to the number line.)

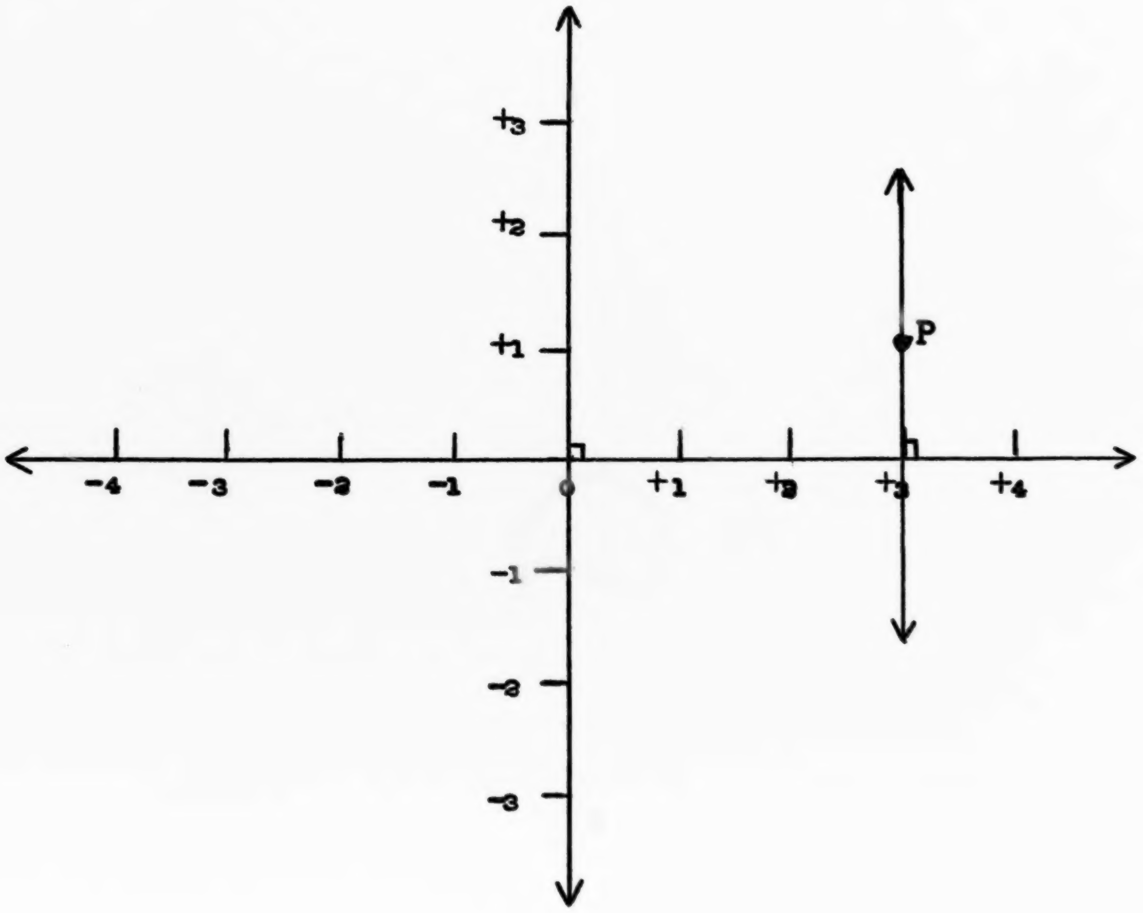


3. Elicit that P is directly above the point associated with +3 on the number line.

Are there any other points in the plane directly above or below the point associated with +3? (all other points on the perpendicular)

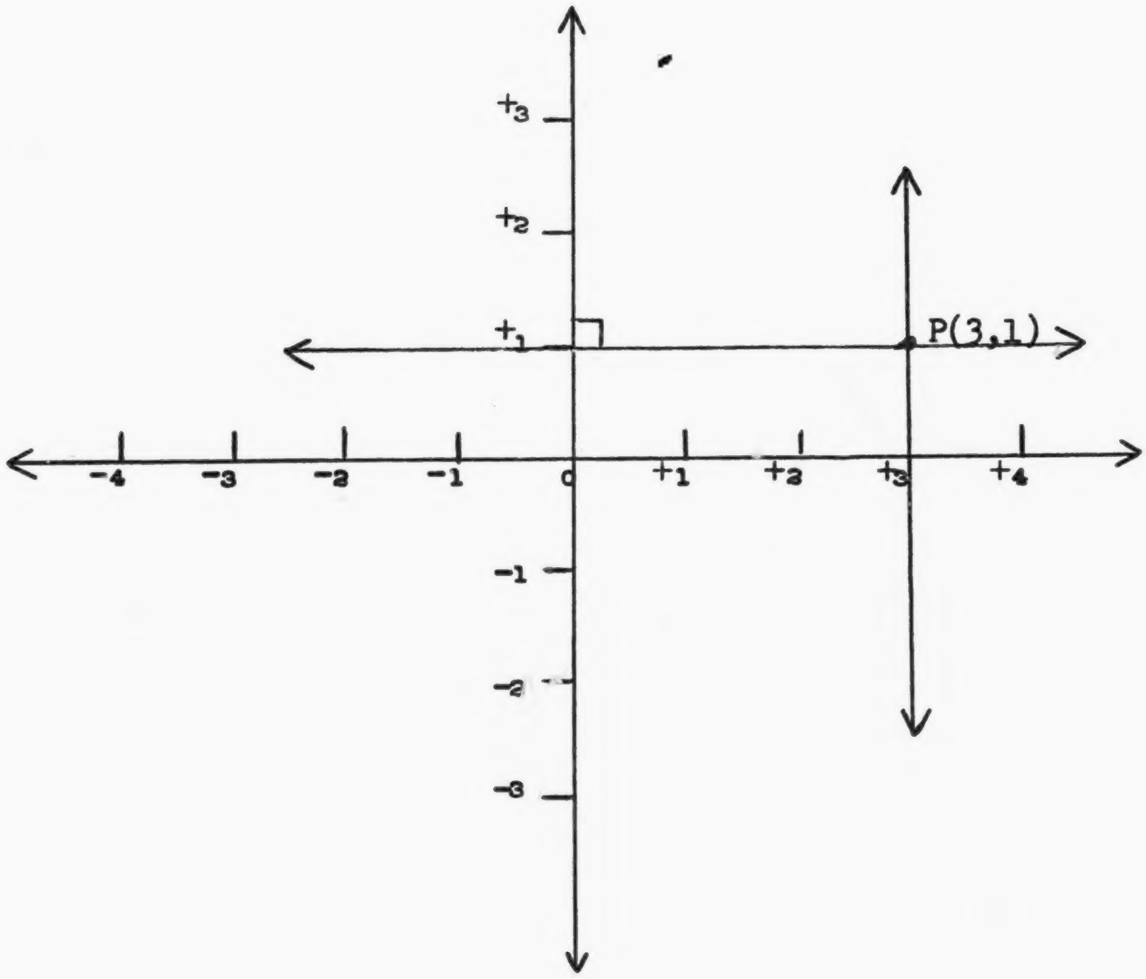
4. How can we distinguish the position of P from all other points on the perpendicular? (by associating numbers with the points on the perpendicular)

   Lead pupils to see that we could also solve this problem by drawing a second number line at right angles to the first in such a way that their zero points coincide.



5. Elicit that P is to the right of some point on this second number line.

   How can we find this point on the second number line? (Draw a line through P perpendicular to the vertical number line.) Point P is to the right of which point on the vertical number line? (the point associated with +1)



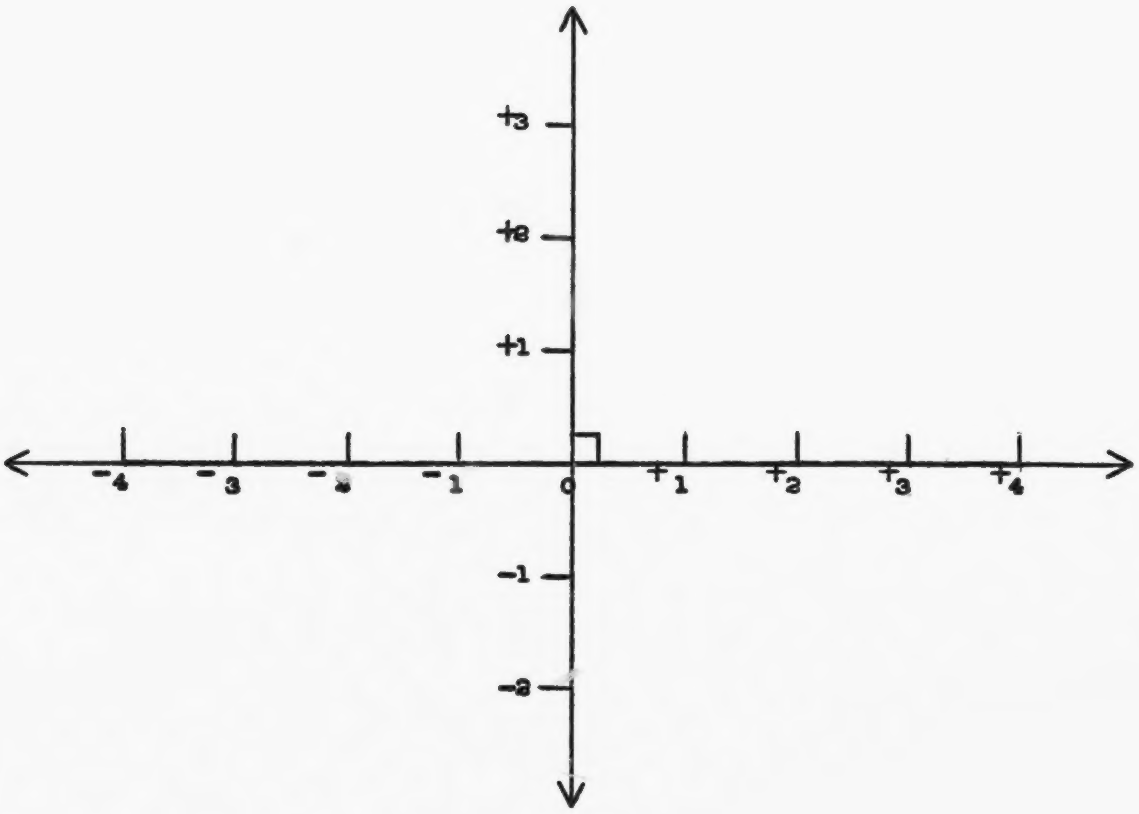
6. Have pupils realize that point P, which is three units to the right of the vertical line and one unit above the horizontal line may be represented by the ordered pair of numbers (3,1). The two numbers assigned to a point are called the coordinates of the point. The first coordinate gives the directed distance of a point from the vertical number line and the second coordinate gives the directed distance from the horizontal number line.

   The plane on which we graph points is called the coordinate plane. The graphing of a point is called plotting a point.

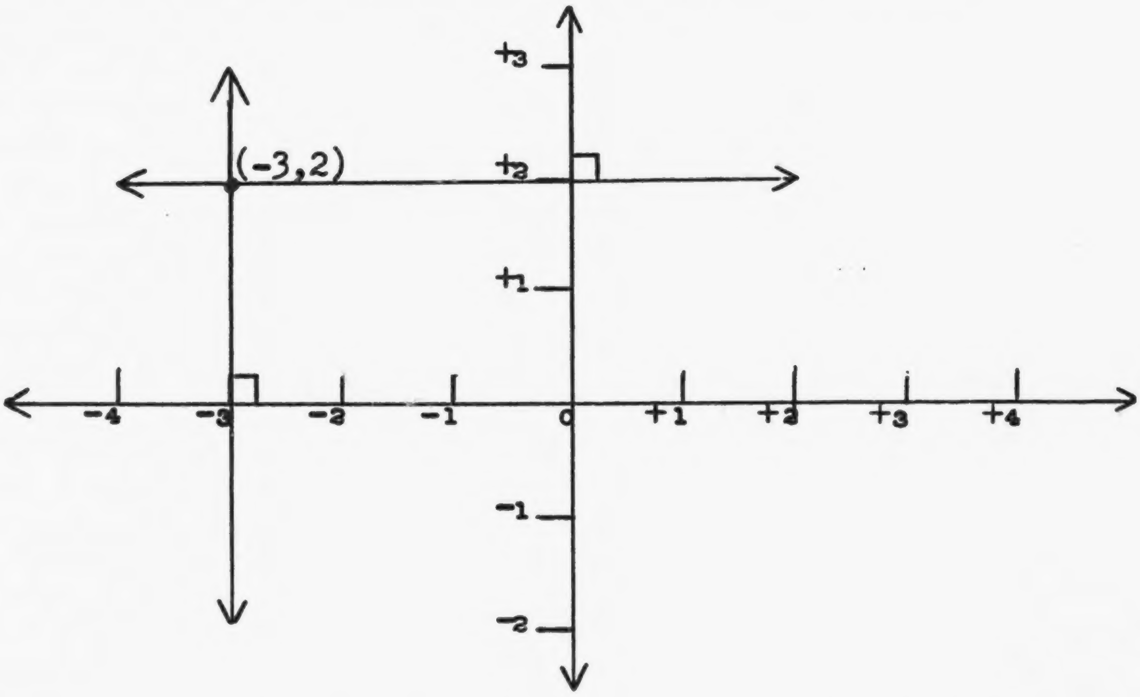
B. To match a pair of numbers with a point in a plane

   1. How can we find the graph of the ordered pair of numbers (-3,2)?

   2. Have pupils draw a vertical number line and a horizontal number line at right angles in such a way that their zero points coincide.



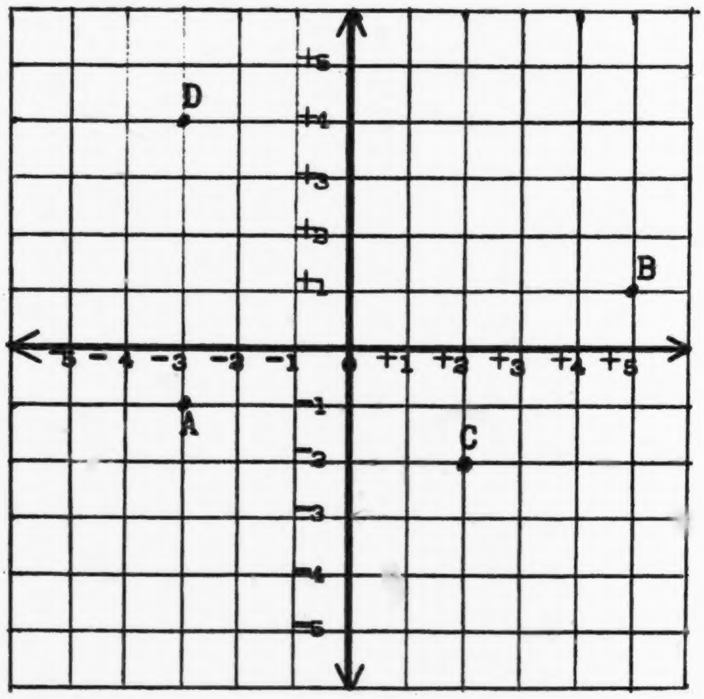
3. To locate the graph of (-3,2), have pupils draw a perpendicular through the coordinate -3 on the horizontal number line, and a perpendicular through the coordinate +2 on the vertical number line. The point of intersection of these lines is the graph of (-3,2).



4. Have pupils locate each of these ordered number pairs on the coordinate plane.

A. (-3,-1) B. (5,1) C. (2,-2) D. (-3,4)

Note to Teacher: After the pupils understand the principle of locating a point in a plane, show them that the use of graph paper will eliminate the necessity for the use of perpendiculars.

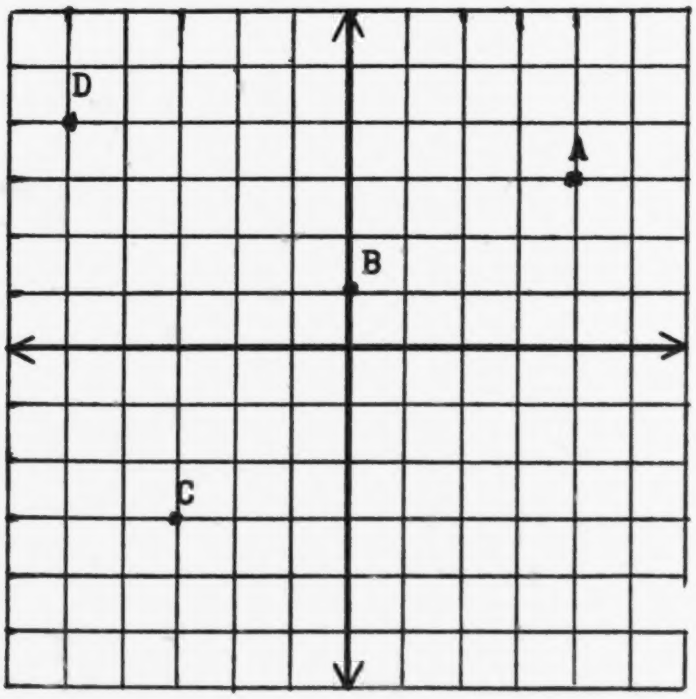

D
B
A
C
+5
+4
+3
+2
+1
-5
-4
-3
-2
-1
0
+1
+2
+3
+4
+5
-1
-2
-3
-4
-5

II. Practice

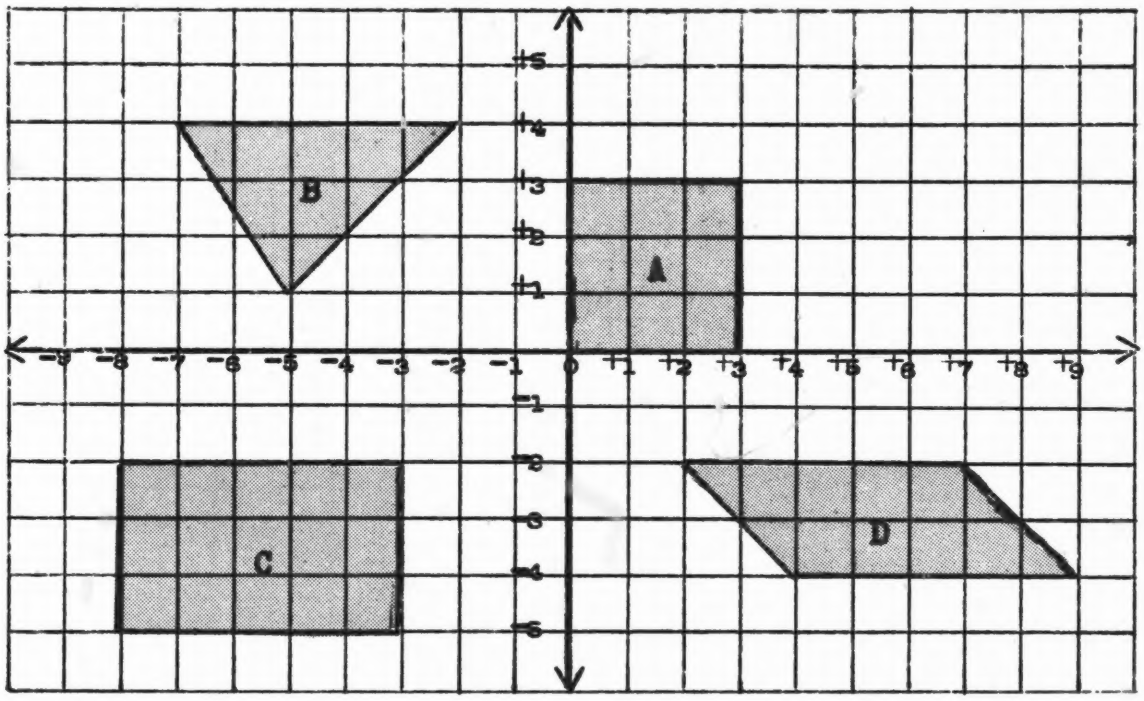
A. Graph each of the ordered number pairs.

1. (1,4)   2. (2,-1)   3. (-3,3)   4. (7,½)

B. Name the coordinates of each point shown below.



C. Name the coordinates of the vertices of each of these polygons.



III. Summary

A. What new vocabulary have you learned today?

B. What are the two numbers in an ordered pair assigned to a point called?

What does the first coordinate indicate? (directed distance of the point from the vertical number line)

What does the second coordinate indicate? (directed distance of the point from the horizontal number line)

Lessons 95 and 96

Topic: Graph of an Equation in Two Variables

Aim: To learn to graph the solution set of an equation in two variables

Specific Objectives:

To review the solution of an equation in two variables
To graph the solution set of an equation in two variables

Challenge: The graph of $3x = 7$ on the number line is a point. What is the graph of $3x + y = 7$ on the coordinate plane?

I. Procedure

A. Review the solution of an equation in two variables

1. Consider the equation: $3x + y = 7$. Find an ordered pair of numbers which will make this sentence a true statement. Remind pupils that we are using the set of real numbers as a replacement set.

   Have pupils find various replacements for x and for y that will result in a true statement. For example,

x	$3x + y = 7$	y
0	$3(0) + y = 7$	7
1	$3(1) + y = 7$	4
2	$3(2) + y = 7$	1

   The ordered pairs of numbers (0,7), (1,4), and (2,1) are solutions of the equation.

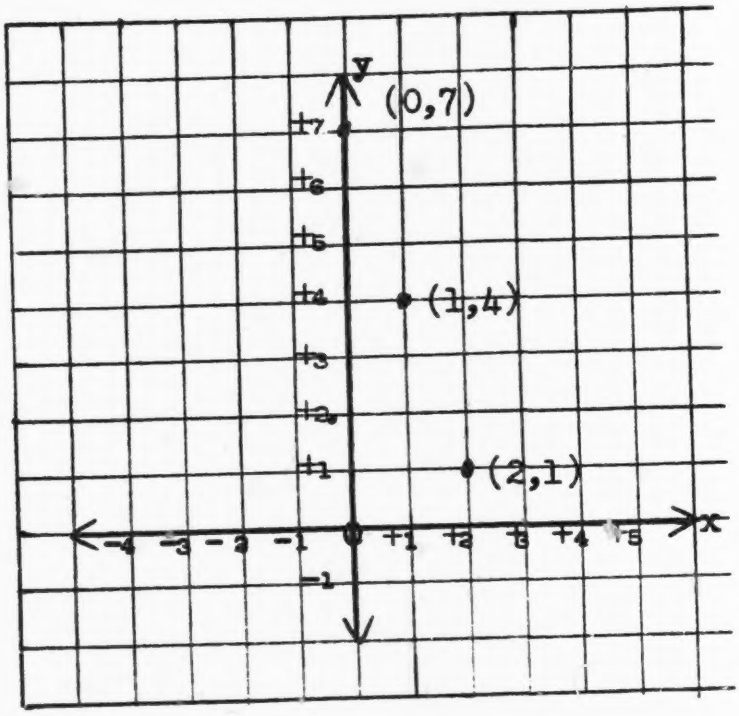
2. Have pupils suggest other ordered pairs of numbers which are solutions of the equation.

3. How many ordered pairs of numbers are there which will make the sentence true? (unlimited number)

4. Tell pupils we can indicate this solution set by using set-builder notation. We write $\{(x,y) \mid 3x + y = 7, x \in R \text{ and } y \in R\}$. This is read: the set of all ordered pairs of numbers, (x,y) such that $3x + y = 7$ and x belongs to the set of real numbers and y belongs to the set of real numbers.

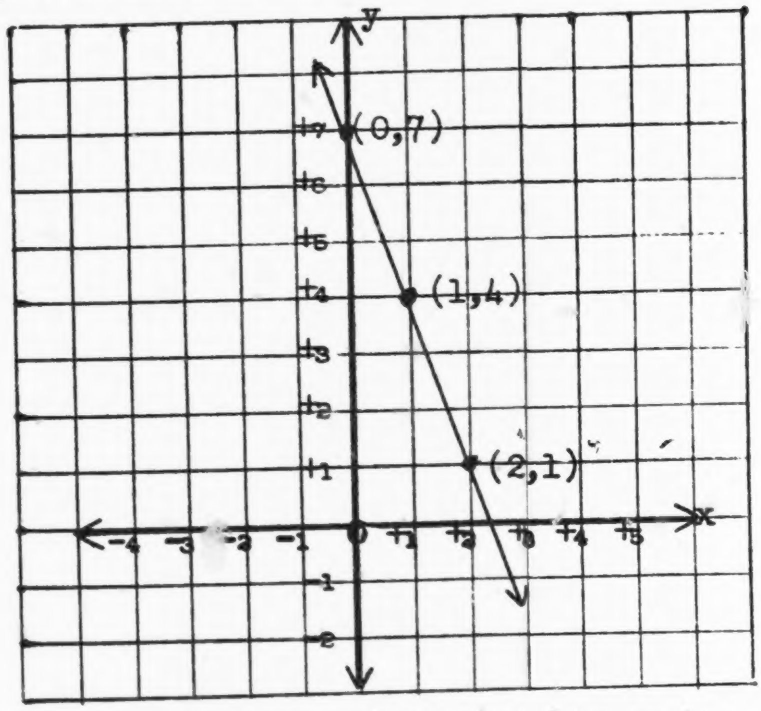
B. Graphing the solution set of an equation in two variables

1. Have pupils plot on the coordinate plane their solutions for $3x + y = 7$.



2. Elicit that the points seem to lie on a straight line.

3. Draw the line.



4. Have pupils choose several other points that lie on the line, as for example (3,-2), (4,-5). Have them test to see that the coordinates of these points are solutions of the equation.

$$3x + y = 7$$
$$3(3) + (-2) \overset{?}{=} 7$$
$$9 + (-2) = 7 \quad \text{True}$$

$$3x + y = 7$$
$$3(4) + (-5) \overset{?}{=} 7$$
$$12 + (-5) = 7 \quad \text{True}$$

5. Have pupils choose a point not on the line, such as (1,1). Are the coordinates of this point a solution of the equation?

$$3x + y = 7$$
$$3(1) + 1 = 7 \quad \text{False}$$

6. Elicit that this line appears to be the set of all those points and only those points whose coordinates satisfy the equation.

   Tell pupils that this is actually so, although we are not ready to prove it mathematically.

   Tell pupils that the line is the graph of the solution set of the equation $3x + y = 7$. This line is also called the graph of the equation.

   Now answer the challenge.

II. Practice

A. Tell whether the given point is on the graph of the equaton.

$x + y = 2$	(1,1)
$x + 2y = 5$	(1,4)
$3x + y = 7$	(-2,3)

B. Graph each of the following equations. (Replacement set is the set of real numbers.)

$$y = 3x + 1$$
$$2x + y = 4$$
$$y = 2 - 3x$$

III. Summary

List the steps in graphing an equation in two variables.

Lessons 97 and 98

Topic: Graph of a Formula

Aim: To learn to graph a formula in two variables

Specific Objectives:

To show the effect of a change in the replacement set on the graph of an open sentence in two variables
To learn to assign an appropriate replacement set for the variables in a formula
Graphing a formula

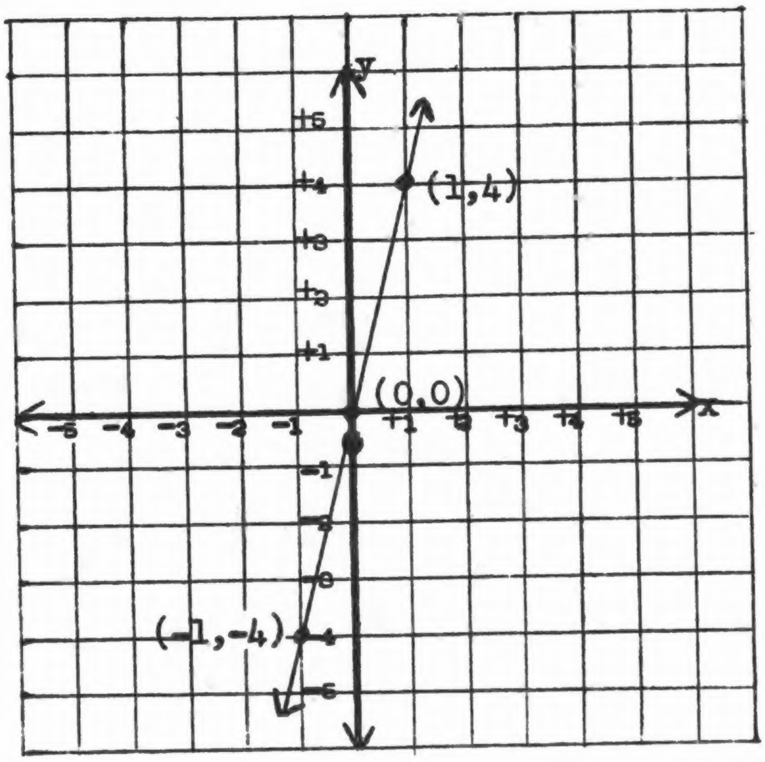
Challenge: Some solutions for the open sentence $y = 4x$ are $(1,4)$, $(-1,-4)$, $(0,0)$.

Which of these are not solutions for the formula of the perimeter of a square: $p = 4s$? Why?

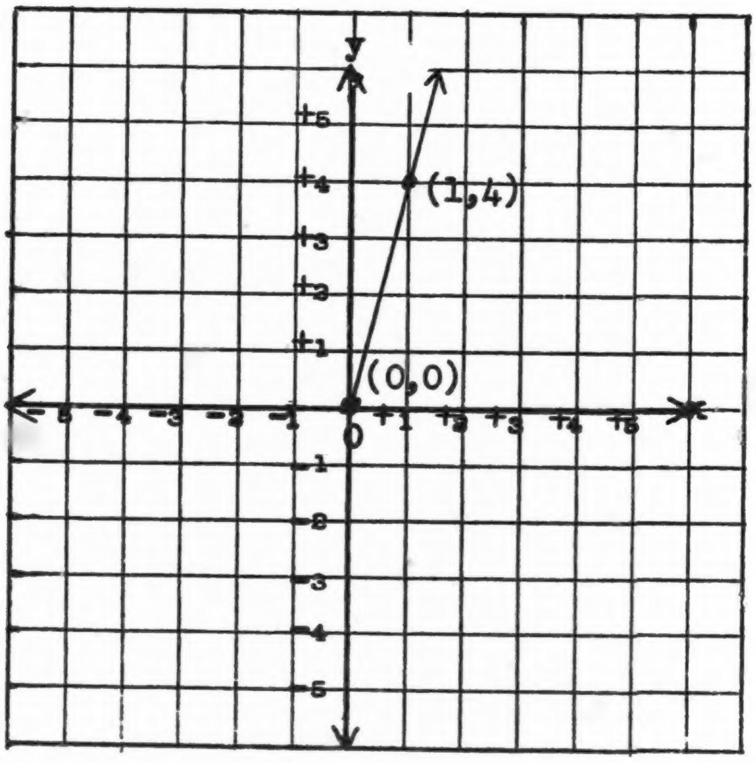
I. Procedure

A. The effect of the replacement set on a graph such as $y = 4x$

1. Consider the equation $y = 4x$. Using the set of real numbers as the replacement set, have pupils graph the equation.



2. Have pupils graph the same equation $y = 4x$ using the set of non-negative real numbers as the replacement set for each variable.



3. What is the name of the geometric figure shown by the set of points in A-1? (line)

   What is the name of the geometric figure shown by the set of points in A-2? (ray)

   Lead pupils to see that because the replacement set contains no negative numbers, this graph is a ray with its end point at (0,0).

4. Have pupils realize that the appearance of a graph changes with a change in the replacement set of each variable.

B. An appropriate replacement set for the variables of a formula

1. Elicit that the formula for the perimeter of a square: $p = 4s$ is similar to the equation $y = 4x$.

2. Since s represents the number of units of measure in the side of a square, why should the value of s not be a negative number?

3. Answer the challenge.

4. Which of the following replacement sets would be appropriate for the variable s in the formula $p = 4s$?

   the set of all numbers you know
   the set of integers
   the set of non-negative real numbers

   Discuss why the last answer is the correct one.

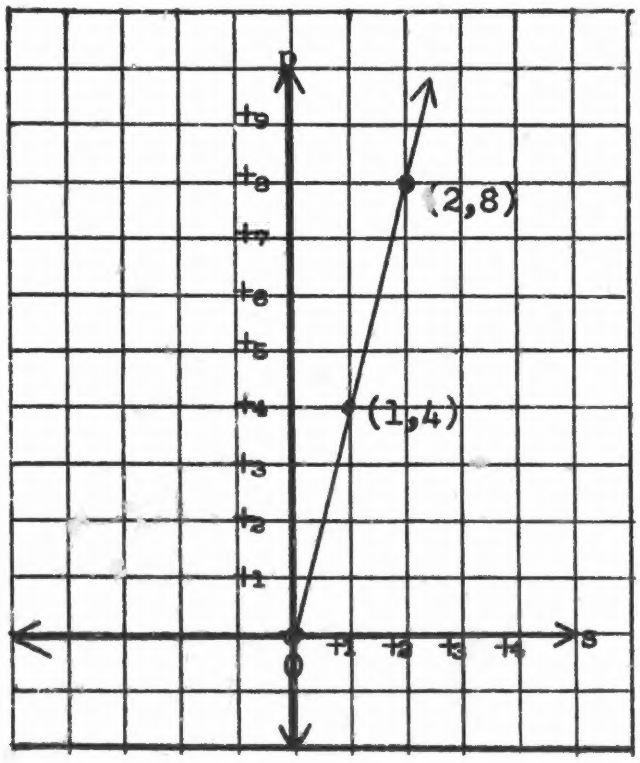
C. Graphing a formula

Have pupils graph $p = 4s$. Replacement set for s: set of non-negative real numbers.

1. Complete the table:

s	$p = 4s$	p
0	$p = 4(0)$	0
1	$p = 4(1)$	4
2	$p = 4(2)$	?
3	?	?

2. Using the ordered pairs, draw the graph:



3. How does this compare with the graph in I-A-2?

4. Elicit that the graph of a formula in two variables is constructed in the same way as the graph of an open sentence in two variables.

4. Why is it necessary in some cases to limit the replacement set for a variable in a formula? If the formula relates to temperature, should the replacement set include negatives?

II. Practice

A. Graph the solution set of the open sentence $y = 3x$. Replacement set for x: set of real numbers.

B. Graph the formula for the perimeter of an equilateral triangle, $p = 3s$. Replacement set for s: set of non-negative real numbers.

C. Discuss the reasons for the difference in the replacement set for the variables in the two graphs.

III. Summary

A. In what way does a change in a replacement set affect the graph of an open sentence?

B. What factor might determine the replacement set for a variable in a formula?

CHAPTER X

SURFACE AREA AND VOLUME

In this chapter are materials and suggested procedures for developing the pupils ability to:

recognize various types of prisms and pyramids
understand the term polyhedron
find the lateral and total surface area of a prism, pyramid, and right circular cylinder
understand and use the formula for the volume of a pyramid and the formula for the volume of a right circular cone

The chapter begins with a brief review of some geometric terms such as closed plane figure, polygon, area, region, closed space figure, and volume. Pupils learn that a polyhedron is a closed space figure, all of whose faces are flat polygonal regions. They then recognize that both the prism and pyramid are polyhedra.

After a brief review of terms applied to space figures such as faces, vertices, and edges, a general method for finding the lateral and total surface area of certain polyhedra is suggested.

The formula for the volume of a prism is reviewed and an experiment suggested by which the pupils discover that the volume of a pyramid is $\frac{1}{3}$ the volume of a prism of congruent base and equal height.

A method is suggested by which pupils discover that a right circular cylinder is composed of a rectangular region and two circular regions. Based upon this understanding, the formula for the surface area of a cylinder is developed.

Pupils review the formula for the volume of a right circular cylinder as preparation for the introduction of the formula for the volume of a right circular cone. Using a procedure similar to the one suggested for finding the volume of a pyramid, pupils are led to realize that the volume of a right circular cone is $\frac{1}{3}$ the volume of a right circular cylinder of congruent base and equal height.

CHAPTER X

SURFACE AREA AND VOLUME

Lessons 99-104

Lesson 99

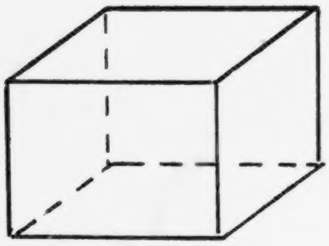
Topic: Plane and Solid Geometric Figures

Aim: To review the characteristics of plane and solid geometric figures

Specific Objectives:

To review the meaning of the terms: plane figure, polygon, region area
To review the meaning of the terms: space figures, solid, prism
To introduce the terms: face, edge, vertices, as applied to space figures

Challenge: How many faces, edges, and vertices does this figure have?



Note to Teacher: It is suggested that in teaching the lessons on space figures in this chapter the teacher use, wherever possible, physical models and permit the pupils to handle them. Some of the lessons require an ability to visualize a space figure from a drawing. However, visualizing a three-dimensional figure from a two-dimensional drawing should come after the pupils have had an opportunity to work with physical models.

It may be necessary to guide pupils in the freehand drawing of three-dimensional figures. Tell pupils that it is customary to represent by line segments only those edges we can see, while the other edges are represented by a series of dashes.

I. Procedure

A. Review of some terms related to plane figures

1. Elicit that a flat surface such as the chalkboard is a model of the mathematical idea called a plane. Unlike the chalkboard, a plane extends indefinitely without limit.
2. Which of the following are models of parts of a plane?

the top of your desk
a piece of chalk

the surface of the floor
a cup

3. Have each pupil select a piece of paper from his notebook.

   a. Elicit that the surface of the paper is a model of part of a plane.

   b. On the paper have pupils draw a picture of a triangle, a rectangle, a pentagon, a hexagon.

   c. Elicit that each of these is a drawing of a closed plane figure.

4. Have pupils recall that any simple closed plane figure formed by the union of line segments is called a polygon.

   a. Elicit that each of the figures they have drawn belongs to the set of polygons.

   b. Have pupils recall that a polygon divides a plane into three sets of points

      those in the interior
      those in the exterior
      those on the polygon (boundary)

      Have pupils illustrate the three sets of points on their drawings.

   c. Elicit that the union of the set of points on the polygon and those in the interior form a polygonal region. A measure of a region is its area.

      Have pupils shade in a region on each of their drawings.

B. Review of some terms related to space figures

1. How does a space figure differ from a plane figure? (A space figure is one all of whose points do not lie in the same plane.)

2. Have pupils point out models of space figures in the classroom, e.g., books, chalk box, and so on. Elicit that the classroom itself is a model of a space figure.

3. Have pupils recall that a closed space figure completely encloses a portion of space just as a closed plane figure encloses a portion of a plane.

4. Elicit that a closed space figure or solid divides space into three sets of points

   those in the interior
   those in the exterior
   those on the figure (boundary)

5. A measure of a solid is its volume.

C. Introducing the terms faces, edges, vertices

1. Show pupils a physical model of the figure in the challenge. Elicit that it is a closed space figure formed by the union of rectangular regions. Have pupils recall that such a figure is a prism.

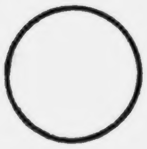
2. Tell pupils that the sides of a prism are called faces. The line segments formed by the intersection of two faces are called edges. The points of intersection of the three faces are called vertices.

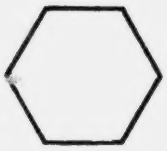
3. Answer the challenge.

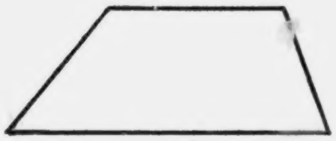
II. Practice

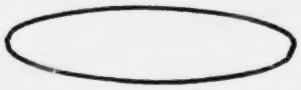
A. Name 3 objects in the classroom which could be considered models of plane figures. Name three models of space figures.

B. Which of these plane figures are polygons?



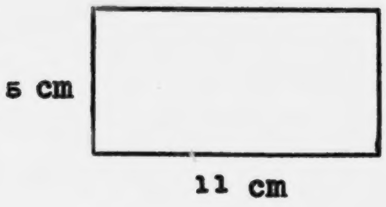






C. Find the area of each of the following regions.

1



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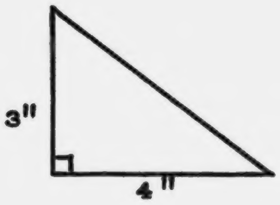

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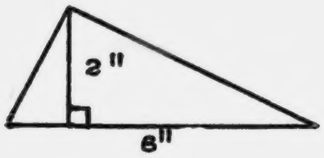


D. How many faces, edges, and vertices does a cube have?

III. Summary

A. Show by illustration the difference between a plane figure and a space figure.
B. Is a polygon a space figure or a plane figure? Name five members of the set of polygons.
C. Use a book as a model and point out a face, an edge, and a vertex.

Lesson 100

Topic: Meaning of a polyhedron

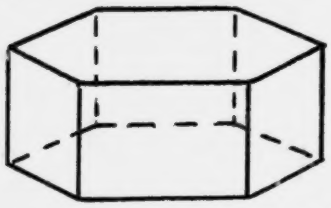
Aim: To learn to recognize various types of polyhedra

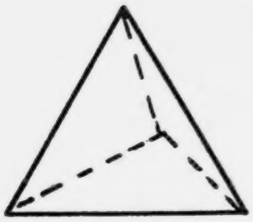
Specific Objectives:

To recognize, name, and sketch various types of prisms and pyramids

To introduce the term polyhedron

Challenge: In what way are these two closed space figures alike?



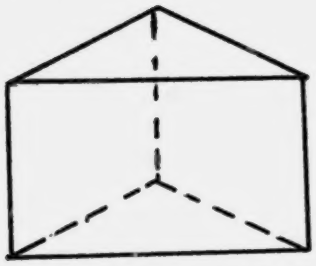


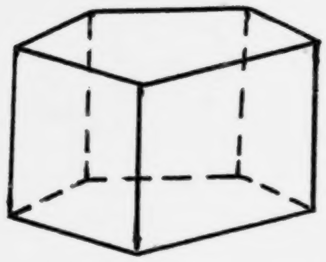
Note to Teacher: In this grade we need not include the oblique prism, oblique pyramid, oblique cylinder, or oblique cone.

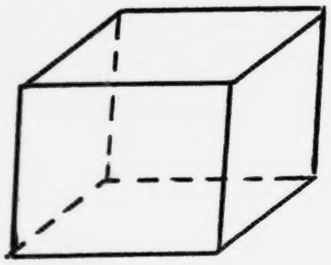
I. Procedure

A. Various kinds of prisms

1. Show pupils models of various types of prisms.







a. Have pupils observe that each prism has at least two parallel congruent faces. These faces are called the bases of the prism.

b. Have pupils observe that a rectangular prism has more than two parallel congruent faces; any such pair may be considered bases.

c. Have pupils observe that the bases of a prism may be any type of polygon region: triangular, rectangular, pentagonal, hexagonal, etc.

d. Tell pupils that a prism is named for the shape of its base. Have pupils identify and sketch models of triangular prism, rectangular prism, etc.

2. a. Faces which are not bases are called lateral faces. Have pupils point out the lateral faces of each prism.

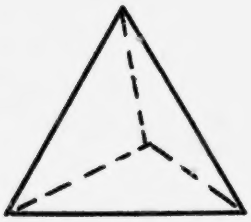
   b. Elicit the number of faces, edges, and vertices for each of the models.

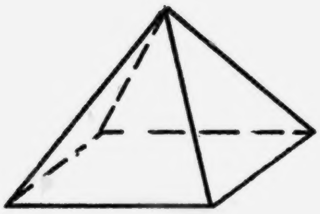
   c. What relation seems to exist between the number of lateral faces a prism has and the number of edges of the polygonal region which forms its base?

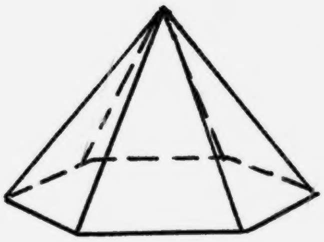
   d. Lead pupils to see that for any of these prisms, regardless of the shape of the base, the lateral faces are always rectangular regions.

   e. Elicit that the common name for a prism, all of whose faces are squares, is a cube.

3. Show pupils models of various types of pyramids.







   a. What type of polygon region is each lateral face of the pyramid? Elicit that all the lateral faces come together in one point. Tell pupils this point is called the apex or vertex of the pyramid.

   b. Name the types of polygons which bound the base of each of the pyramids.

   c. Tell pupils that the pyramid, like the prism, takes its name from the shape of its base. Have pupils identify and sketch models which represent a triangular pyramid, rectangular pyramid, pentagonal pyramid, hexagonal pyramid, etc.

   d. For each model of a pyramid, elicit the number of faces, the number of edges, the number of vertices.

B. The Polyhedron

1. Have pupils compare the models of the set of prisms with the set of pyramids. Elicit that each is a closed space figure, each is composed of flat (plane) surfaces which are polygonal regions.

2. Answer the challenge.

3. Tell pupils that the name polyhedron is given to any closed space figure formed by the union of plane surfaces which are polygonal regions. The word polyhedron comes from the Greek: "poly" meaning "many" and "hedra" meaning "faces." Elicit that a prism and a pyramid are members of the set of polyhedra (plural of polyhedron).

II. Practice

A. Name and sketch three objects in the classroom which have the shape of a prism.

B. Name any three objects which have the shape of a pyramid. Sketch these.

C. What is the difference between the number of faces, edges, and vertices of a cube and a square pyramid?

D. How many faces would an octagonal prism have? an octagonal pyramid?

E. (Optional) Have pupils count the number of faces, edges, and vertices of several prisms and pyramids. Have them observe that when F represents the number of faces, E the number of edges, and V the number of vertices, then $F - E + V = 2$ for all polyhedra studied so far.

III. Summary

A. How is the name of a prism determined?
How is the name of a pyramid determined?

B. What do we mean when we say a figure is a polyhedron?

Lesson 101

Topic: Surface Area

Aim: To develop a method for finding the surface area of a prism and of a pyramid

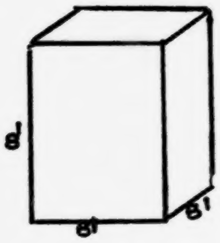
Specific Objectives:

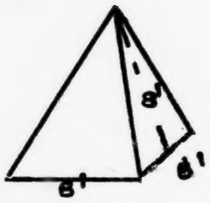
To learn the meaning of surface area of a polyhedron

To develop a method for finding the surface area of a prism

To develop a method for finding the surface area of a pyramid

Challenge: Jack is to paint the total surface of this prism. Joel is to paint the total surface of this pyramid. Which boy has the greater area of surface to paint? State the difference in area in square feet.





I. Procedure

A. Refer to challenge.

1. Elicit that to answer the challenge we find the total surface area of each figure. We then subtract to find the difference in area.

2. Consider the prism. What kind of polygonal region is each face?

3. What is the area of the top face? Which face has an area equivalent to the area of the top face? What is the combined area of these two faces? (72 sq. ft.)

4. What is the area of the face whose length is 6' and height is 8'? (48 sq. ft.) Have pupils indicate the other faces with an area of 48 sq. ft.

5. Total area of the four lateral faces is 4 x 48 or 192.

6. Total area = 2 (area of base) + area of the lateral faces
   = 72 + 192
   = 264

The total surface area of the prism is 264 sq. ft.

B. Have pupils study the model of the pyramid.

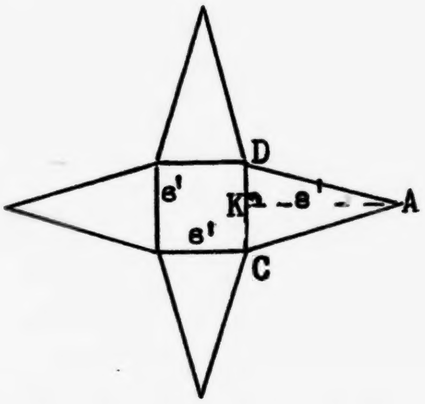
1. What type of polygonal region is the base of the pyramid? (square region)

2. What type of region is each lateral face? (triangular region)

3. Elicit that to find the area of a triangular region the base and height must be known.

4. Refer to diagram in challenge. Show pupils how the pyramid would look if it were cut along the lateral edges so that it would lie flat.

   Elicit that the line segment $\overline{AK}$ is the altitude of the triangle CDA and its length is 8'.



5. Tell pupils that while the length of $\overline{AK}$ is the height of the triangle CDA, it is called the slant height of the pyramid. Elicit that the name slant height is an appropriate one.

6. Elicit that the height of each triangular region is the same in this case.

7. What is the area of the triangular region CDA? ($\frac{1}{2}(6\times8)$) How many such regions are there? What is the total lateral area? (4×24 or 96 sq. in.)

8. What is the area of the base region? (36 sq. in.)

9. To find the total surface area, we find the sum of the areas of all the faces.

   Total area = area of base + area of lateral faces
   = 36 + 4 (24)
   = 36 + 96
   = 132

   The total surface area of the pyramid is 132 sq. ft.

10. Answer the challenge.

    Jack has to paint 264 sq. feet.
    Joel has to paint $\underline{132}$ sq. feet.
    Difference 132 sq. feet.

II. Practice

A. Find the total surface area of a cube if one edge is 9 inches.

B. Find the total surface area of a square pyramid if one edge of the base is 4 inches and the slant height is 8 inches.

C. Find the lateral surface area of a regular prism if its base is 7 inches long, 8 inches wide, and its height measures 10 inches.

III. Summary

A. How do we find the total surface area of a prism?

B. How do we find the total surface area of a pyramid?

C. What is the meaning of slant height?

D. What is the difference between total surface area and lateral surface area?

Lesson 102

Topic: Volume of a Pyramid

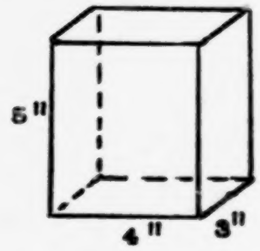
Aim: To develop a formula for finding the volume of a pyramid

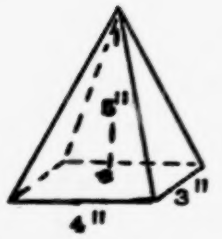
Specific Objectives:

To review the formula for finding the volume of a prism

To relate the formula for the volume of a pyramid to the formula for the volume of a prism

Challenge: How many times as great as the volume of the pyramid is the volume of the prism?





I. Procedure

A. Review the formula for the volume of a prism.

1. Refer to figure in challenge. Elicit that the volume of a prism can be found by the formula $V = Bh$ where B represents the area of the base and h the height of the prism.

2. Have pupils find B.

$$B = l \times w$$
$$= 4 \times 3$$
$$= 12$$

$$V = Bh$$
$$= 12 \times 5$$
$$= 60$$

The volume of the prism is 60 cu. in.

B. Volume of a pyramid

1. Refer to the pyramid in the challenge. Tell pupils that the height of a pyramid is the length of the perpendicular line segment from the apex to the base.

$\overline{EF}$ is the perpendicular from the apex A to the base region ADCB. The height of the pyramid is the measure of $\overline{EF}$.

Note to Teacher: Using models of a prism and a pyramid whose bases are congruent and whose heights are the same, it can be shown by experimentation that the volume of a pyramid is $\frac{1}{3}$ the volume of a prism.

By filling the pyramid with salt and emptying it into the prism, the pupils will see that the prism contains 3 times the amount of salt contained in the pyramid.

In presenting the lesson, use the experiment above if models are available. If not, tell pupils it can be done.

2. The formula for the volume of a prism is $V = Bh$; therefore, the formula for the volume of a pyramid is $V = \frac{1}{3} Bh$.

3. Have pupils find the volume of the pyramid in the challenge.

$$\begin{aligned} V &= \frac{1}{3} Bh \\ &= \frac{1}{3} \times 12 \times 5 \\ &= 20 \end{aligned}$$

Volume of the pyramid is 20 cu. in.

4. Answer the challenge.

II. Practice

A. Find the volume of a pyramid with the following bases and heights.

Area of Base	Height
10 sq. in.	3 in.
15 sq. in.	2 in.
25 sq. in.	6 in.

B. If the area of the base of a prism is 30 sq. in. and its height is 10 inches, find the volume of a pyramid with the same height and congruent base.

C. If the area of the base of a pyramid is 8 sq. ft. and its height is 3 feet, what is the volume of a prism with the same height and a congruent base?

D. Find the volume of a pyramid whose base is a square, the measure of whose side in inches is 8, and whose height is 12 inches.

E. The area of the base of a pyramid is 81 square inches. The height of the pyramid is 6 inches. Find the volume of the pyramid.

III. Summary

A. What is the general formula for the volume of a prism?

B. How does the volume of a prism compare with the volume of a pyramid of the same height and base?

C. How does the volume of a pyramid compare with the volume of a prism of the same height and base?

Lesson 103

Topic: Surface Area of a Right Circular Cylinder

Aim: To develop a method for finding the surface area of a right circular cylinder

Specific Objectives:

To learn the meaning of surface area of a right circular cylinder
To develop a formula for finding the lateral area of a right circular cylinder
To develop a formula for finding the total surface area of a right circular cylinder

Challenge: What two familiar geometric figures are combined to form the surface of a right circular cylinder?

I. Procedure

A. Meaning of surface and surface area of a cylinder

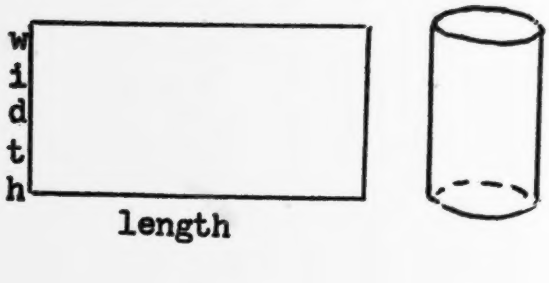
1. Show pupils models of right circular cylinders. If necessary, use a can of soup, a cereal container, etc.

   What kind of geometric figure is each of the two bases? (circular regions)

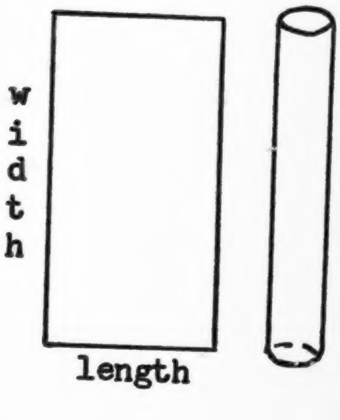
   How do the bases compare in size?

   What word is used to describe two geometric figures which have the same size and shape?

2. Show pupils how a cylindrical surface can be formed from a rectangular sheet of paper.



or



3. Answer the challenge.

B. Lateral area of a right circular cylinder

1. Using the models in A-2, elicit the following:

   a. how the circumference of the circular base of the cylinder compares with the length of the paper (They are the same.)

   b. how the height of the cylinder compares with the width of the paper (They are the same.)

2. What formula can be used to find the area of a rectangular region? ($A = \ell \times w$)

3. Elicit that the formula, $A = \ell \times w$, would be written in terms of the circumference of the base of the cylinder and its height.

$$\begin{array}{rcccc} A & = & \ell & \times & w \\ & & \downarrow & & \downarrow \\ A & = & C & \times & h \end{array}$$

4. Have pupils recall the formula for the circumference of a circle $C = 2\pi r$.

   Replacing C with $2\pi r$ in the formula, we arrive at the formula for the lateral surface of the cylinder. $A = 2\pi rh$.

C. Total surface area of a right circular cylinder

1. Elicit that the total surface area is the sum of the lateral area and the area of the two bases.

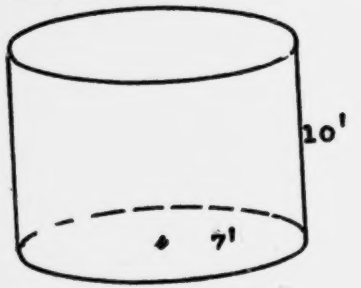
2. What formula is used to find the area of a circular region? ($A = \pi r^2$)

3. Elicit that the bases are congruent and have the same area.

4. Total area = lateral area + 2 (area of base)

$$\begin{array}{rll} \text{Total area} & = \text{lateral area} & + \ 2\,(\text{area of base}) \\ & = 2\pi rh & + \ 2\pi r^2 \end{array}$$

5. Use $\pi = \frac{22}{7}$ to find the total surface area of this cylinder.



$$\begin{array}{rll} \text{Total area} & = 2\pi rh & + \ 2\pi r^2 \\ & = 2\times\frac{22}{7}\times 7\times 10 & + \ 2\times\frac{22}{7}\times 7\times 7 \\ & = 440 & + \ 308 \\ & = 748 & \end{array}$$

Total surface area of the cylinder is 748 sq. in.

II. Practice

A. Name 3 items which could serve as models of a right circular cylinder.

B. Find the lateral area of a right circular cylinder whose radius measures 14 inches and whose height is 8 inches. ($\pi = \frac{22}{7}$)

   1. Find the area of the bases.

   2. Find the total surface area of the cylinder.

III. Summary

A. What areas are included in finding the total surface area of a right circular cylinder?

B. Explain the meaning of the formula: lateral area = $2\pi rh$.

C. Explain the meaning of the formula: total area = $2\pi rh + 2\pi r^2$.

Lesson 104

Topic: Volume of a Right Circular Cone

Aim: To develop the formula for the volume of a right circular cone

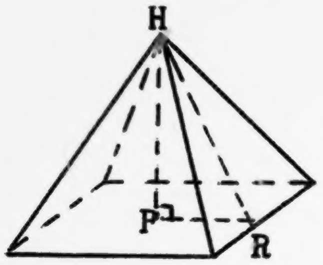
Specific Objectives:

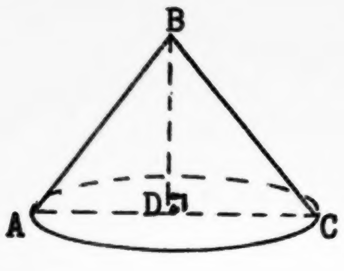
To recognize some similarities and differences between a rectangular pyramid and a right circular cone
To review the formula for finding the volume of a right circular cylinder
To develop the formula for the volume of a cone

Challenge: A rectangular pyramid and a right circular cone have some features in common and some which are different. Name some common features. Name one characteristic of a pyramid that is not a characteristic of a cone.

I. Procedure

A. Comparing the pyramid and the cone





Refer to challenge.

1. Elicit that both figures are closed space figures and the base of the pyramid and the base of the cone are plane geometric regions. (rectangular region, circular region)

2. Elicit that the base of a circular cone must be bounded by a circle, but the base of a pyramid may be any type of polygonal region.

3. Name the line segment which is the altitude of the pyramid and the altitude of the cone.

4. Elicit that the length of $\overline{HR}$ is the slant height of the pyramid. Which line segment do you think should be measured to find the slant height of the cone? ($\overline{BC}$ or $\overline{AB}$)

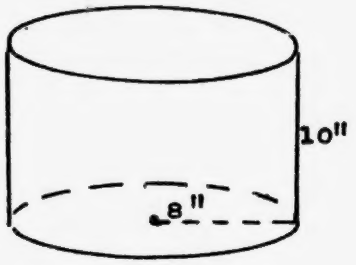
5. Name the apex (vertex) of the pyramid. What point on the cone corresponds to the apex of the pyramid?

6. Answer the challenge.

B. Review formula for volume of right circular cylinder

1. Have pupils recall that $V = \pi r^2 h$ is the formula for the volume of a cylinder, where πr^2 refers to the area of the circular base region, and h to the height of the cylinder.

2. Find the volume of this cylinder to the nearest unit. Use $\pi = 3.4$.



$V = \pi r^2 h$
$= 3.14 \times 64 \times 10$
$= 200.96$

The volume to the nearest unit is 201 cu. inches.

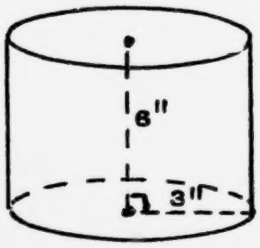
C. To develop the formula for a right circular cone

1. As was done in arriving at the formula for the volume of a pyramid, show by use of models that a cylinder will hold 3 times the amount of salt as a cone of equal height and congruent base.

2. Have pupils realize that just as the formula for the volume of a pyramid is $\frac{1}{3}$ the volume of a prism, the formula for the volume of a cone is $\frac{1}{3}$ the volume of a cylinder.

Volume of a cylinder $= \pi r^2 h$
Volume of a cone $= \frac{1}{3} \pi r^2 h$

3. Find the volume to the nearest whole unit. Use $\pi = 3.14$.

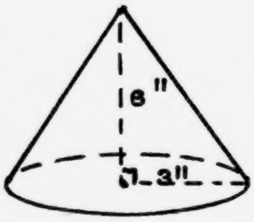


$V = \pi r^2 h$

$V = 3.14 \times 9 \times 6$

$V = 169.56$

Volume to nearest unit
is 170 cu. in.



$V = \frac{1}{3} \pi r^2 h$

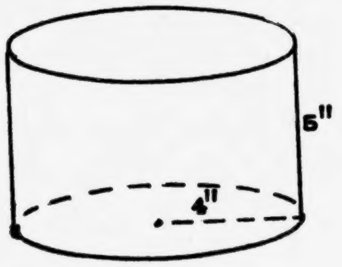
$V = \frac{1}{3} \times 3.14 \times 9 \times 6$

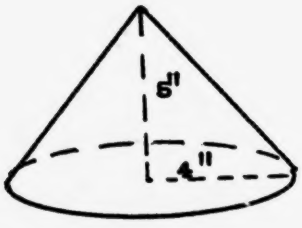
$V = 56.52$

Volume to nearest unit
is 57 cu. in.

II. Practice

A. Find the volume to the nearest unit for each of the following. (Use $\pi = 3.14$)





B. Find the volume of each of these right circular cones. Use either $\pi = \frac{22}{7}$ or $\pi = 3.14$, whichever will make your computations easier.

Radius of base	Height
7"	10"
10"	7"
5 cm	12 cm

III. Summary

A. Name some common characteristics of a rectangular pyramid and a right circular cone.

B. Name one difference between a rectangular pyramid and a right circular cone.

C. Is a right circular cone a polyhedron? Why?

D. What relationship exists between the volume of a cone and the volume of a right circular cylinder of equal base and height?

CHAPTER XI

STATISTICS AND PROBABILITY

The first part of the material in this chapter is concerned with developing an understanding of statistics as a systematic collection, organization, and analysis of data. Pupils are helped to appreciate the use of a measure of central tendency as the most representative score of a set of scores.

The lessons included in this section will help pupils

organize data
interpret histograms and frequency polygons
find the mean, mode, and median of a set of data
understand the use of percentile rank
understand the meaning of probability
interpret a probability ratio
compute a probability ratio
understand the meaning of a sample space
represent the probability of an event as a per cent

The beginning lesson of the chapter investigates the need for a method of organizing a large set of data. Procedures are suggested for the construction of frequency distribution tables. These tables are used to find the relative frequency of a score.

Through the use of histograms and frequency polygons, pupils are helped to visualize a frequency distribution. They are led to make use of their previous knowledge of bar graphs and line graphs to read and interpret histograms and frequency polygons.

The meaning of a measure of central tendency as a score which describes a special feature of a set of data is developed through the computation and interpretation of mean, mode, and median. Pupils should be guided to use the most appropriate measure of central tendency when drawing conclusions from a set of data.

The concept of percentile rank is developed because of its increasing use in ranking students for educational purposes. The pupils are led to an understanding of percentile rank as an indication of a pupil's position in a group. This is shown by computing the per cent of pupils in the group who rank below him.

In the second part of the chapter some concepts in the field of probability are introduced.

Probability had its beginning in the attempts of gamblers to persuade mathematicians to predict their chances of winning. Today, it has taken its place among the important branches of mathematics because of its use in research, science, industry, government, and education.

Only the simpler concepts of probability are included. The topic is introduced by eliciting from the pupils that some things cannot happen, some things are certain to happen, some things may happen. A probability of 0 is assigned to an event that cannot happen, and a probability of 1 is assigned to an event that is certain to happen. The probability of an event P(E) that may happen is measured by a ratio such that: $0 \leq P(E) \leq 1$.

Through the use of simple experimental situations, pupils learn the meaning of and the use of the probability ratio in forecasting the likelihood of a simple event taking place. The concept of a sample space as the set of all possible outcomes of an event is then developed and used to find the probability ratio in problems involving two distinguishable events taking place at the same time. Finally, representing the probability ratio as a per cent has been introduced because of the tendency today of forecasting the possibility of rain or snow in terms of a per cent.

CHAPTER XI

STATISTICS AND PROBABILITY

Lessons 105-115

Lesson 105

Topic: Statistics

Aim: To organize data into a frequency distribution table

Specific Objectives:

To discover the advantage of organizing a large set of data
To construct a frequency distribution table for a set of data

Challenge: If Morris is 61" tall, is he one of the tallest or one of the shortest boys in his class?

I. Procedure

A. Organizing data

1. Refer to challenge. Elicit that they cannot answer the question until they know the heights of the other members of the class. Discuss the importance of not jumping to conclusions without sufficient data. Tell pupils that any set of numerical facts can be called data.

2. Suppose the heights of the pupils in Morris' class are as follows:

5'6"
5'2"
65"
4'8"
5'6"
5'0"
66"
5'2"
71"
5'4"
58"
5'6"
5'1"
64"
66"
4'7"
68"
61"
5'6"
5'9"

3. Can the challenge question now be easily answered?

Elicit that an orderly arrangement of data would prove helpful in finding the answer to the challenge. Tell pupils that each of the numerical facts can be called a score.

a. How may we arrange the scores to more easily compare the heights? (Arrange in order of size, beginning with shortest or tallest.) Placing scores in order of size is called ranking the scores.

b. The difference between the greatest and the least score is called the range. The range for this set of data is 71-55 or 16.

c. How do we compare measures that are expressed in different units? (Change all measures to same unit.)

d. How would you express 5'6" in inches? (66")

B. Construction of a frequency distribution table

1. Help the pupils make a table which shows the frequency of each height. Tell the pupils that such a table is called a frequency distribution table.

HEIGHTS OF PUPILS IN MORRIS' CLASS

Height in Inches	Tally	Frequency
55	/	1
58	/	1
60	//	2
61	/	1
62	//	2
64	//	2
65	/	1
66	~~////~~ /	6
68	//	2
69	/	1
71	/	1

2. From the table, have pupils answer such questions as:

a. How many pupils are 55" tall? 61" tall? 59" tall? How many pupils are more than 62" tall? less than 58" tall?

b. Answer the challenge question.

c. What is the sum of the frequencies of all scores? (20) What is the ratio of the number of pupils who are 64" tall to the number of pupils in the group? ($\frac{2}{20}$ or $\frac{1}{10}$)

How can you express this as a per cent? as a decimal? (10%, .10)

Tell pupils that the ratio of the frequency of a score to the sum of all the frequencies in a set of data is called the relative frequency of the score.

d. How many pupils are at least 66" tall? Lead pupils to see that all pupils less than 66" tall (4) and those who are 66" tall (6) must be included in the answer. The number of pupils at least 66" tall is 10.

e. How many pupils are between 55" tall and 58" tall? Pupils should realize that in this case neither those 55" tall nor those 58" tall should be included.

3. Have the pupils realize that because of the organization afforded by the table, it was easier to analyze the data.

4. Tell pupils that the systematic collection, organization, and analysis of data is a branch of applied mathematics called Statistics.

II. Practice

A. Make a frequency distribution table for the following set of test scores made by members of class 8-205:

75, 80, 85, 55, 60, 75, 60, 50, 55, 85, 90, 60, 65, 80, 55,
80, 70, 85, 85, 60, 75, 95, 85, 50, 55, 75, 65, 85, 90, 75

If the passing mark is 65, how many pupils passed? How many failed?

What per cent of the class failed? What per cent of the class passed?

How many pupils received a mark greater than 75? a mark of at least 75?

B. Make a frequency table showing the frequencies of the vowels contained in the following sentence: A frequency table is a method used to organize data.

C. Which vowel is used the least number of times? the greatest number of times? Is there a vowel that is not used in the sentence?

III. Summary

A. Why is it advantageous to organize data before making conclusions?
B. What is meant by a frequency distribution table?
C. Name the headings used in constructing a frequency table.
D. Name the steps in organizing the data in a frequency distribution table.
E. What new vocabulary have you learned today?

Lesson 106

Note to Teacher: It is suggested that where an overhead projector is available, a series of transparencies be prepared of the histograms and the frequency polygons presented here. Where such a projector is not available, it is suggested that rexograph sheets be prepared for pupil use.

Topic: Statistics

Aim: To learn to interpret a histogram and a frequency polygon

Specific Objectives:

To show how the data in a frequency distribution is represented by a graph called a histogram
To read and interpret data presented by a histogram
To read and interpret data presented by a frequency polygon

Challenge: A set of data is reported in newspapers, magazines, and textbooks by means of a paragraph, a table, or a graph. Which one do you prefer? Why?

I. Procedure

A. Representing data given in a frequency distribution by a graph

1. Answer the challenge. Elicit that these are three ways of reporting data and each tells approximately the same story, but that the graph tells the story more dramatically.

2. Have pupils consider the following set of test scores made by the members of class 8-324.

75, 83, 55, 60, 75, 62, 51, 84, 90, 60, 63, 77, 83, 70
85, 84, 60, 75, 95, 86, 72, 76, 68, 84, 89, 76, 58

a. Elicit that some sets of scores would require a very long frequency distribution table. Suggest that if the scores were grouped in intervals of ten, the table could be shortened.

b. Help the pupils arrange the scores in intervals of ten.

c. Have pupils prepare a table using the headings: Score Intervals, Tally, Frequency.

Score Intervals	Tally	Frequency
91-100	/	1
81-90	𝍸 ////	9
71-80	𝍸 //	7
61-70	////	4
51-60	𝍸 /	6
Below 51		0

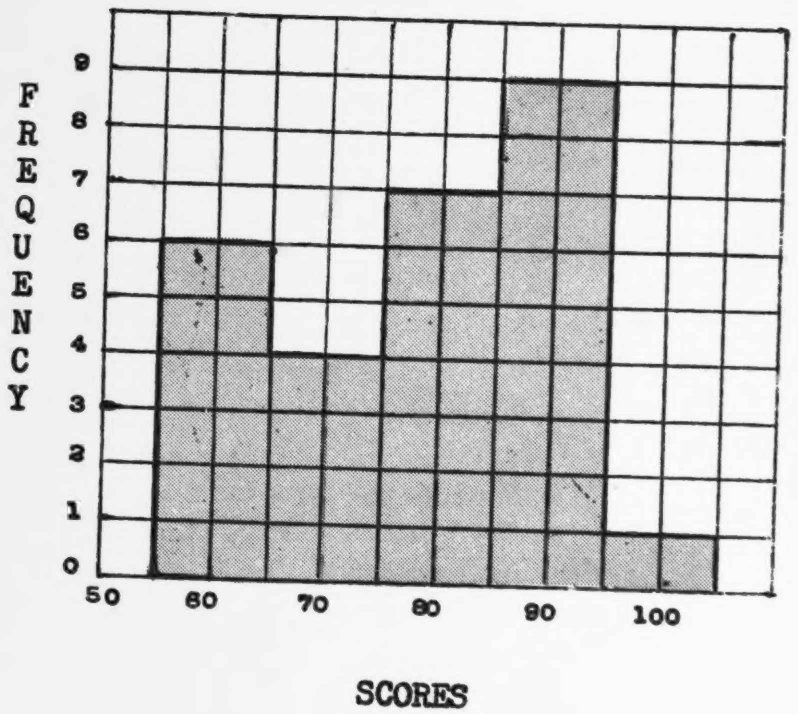
d. Do you think this information could be presented in the form of a graph?

3. Present Chart I

Tell pupils that this frequency distribution can be pictured by means of a special type of bar graph called a histogram.

Chart I — TEST SCORES

HISTOGRAM — Class 8-324



B. Reading and interpreting a histogram
From the histogram, have the pupils answer the following questions:

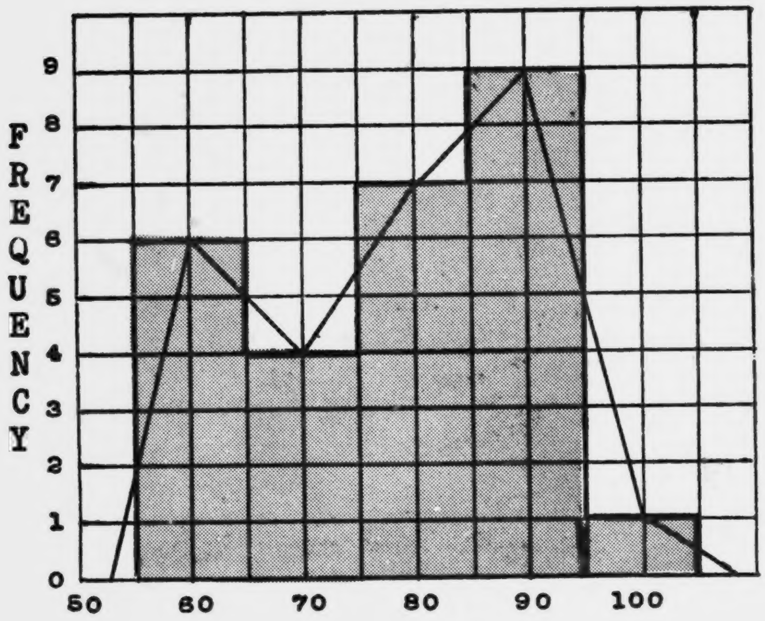
1. What is shown on the vertical axis?

2. What is shown on the horizontal axis?
3. Which interval has a frequency of 7?
4. Which interval has a frequency of 1?
5. Which intervals have a frequency of zero? (all intervals except those from 51 to 100)
6. What type of graph does a histogram resemble?

C. Reading and interpreting a frequency polygon

Present Chart II - Histogram and Frequency Polygon

Chart II HISTOGRAM AND FREQUENCY POLYGON



SCORES

1. Show the pupils that if the midpoints of the tops of adjacent bars are connected by line segments, we have a special type of broken line graph.

2. Elicit that the line segments joining the midpoints of the bars and the horizontal axis form a polygon. (A polygon is a simple closed curve formed by the union of line segments.)

   Tell pupils that this type of graph of the frequency distribution is called a frequency polygon.

3. Tell pupils that a frequency polygon can be made without first making a histogram by plotting the middle score of each interval.

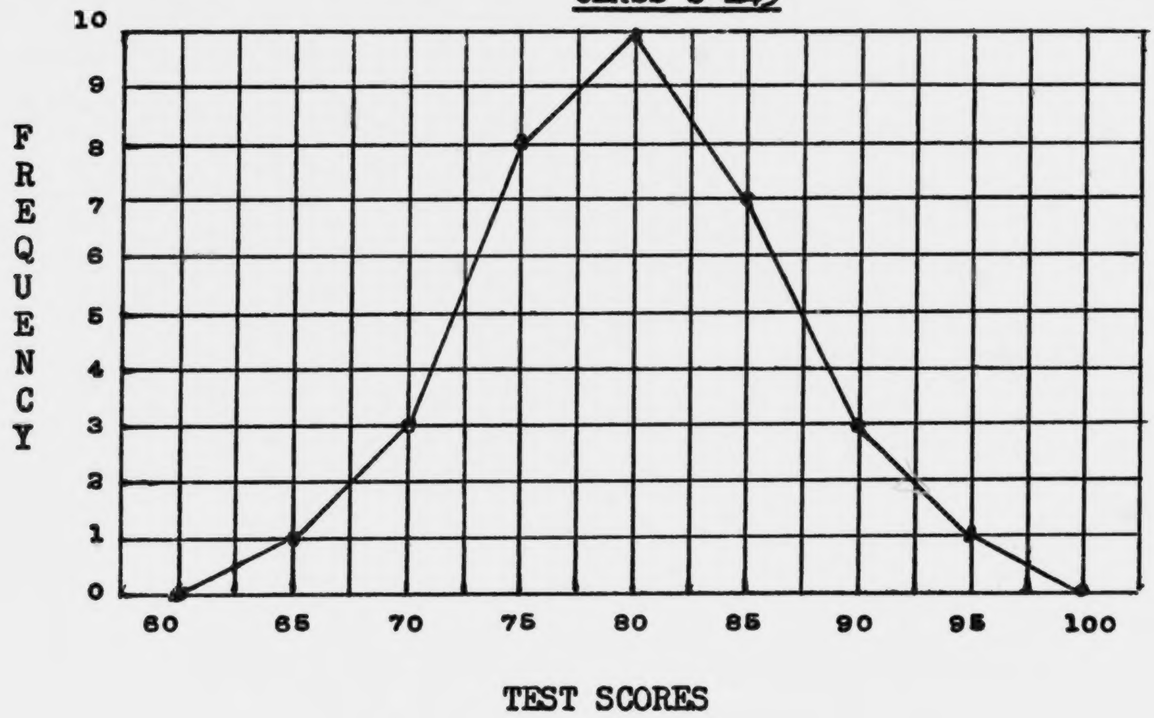
Consider the following frequency table:

Score Intervals	Tally	Frequency
Above 97		0
93-97	/	1
88-92	///	3
83-87	𝍸 //	7
78-82	𝍸 𝍸	10
73-77	𝍸 ///	8
68-72	///	3
63-67	/	1
Below 63		0

4. Present Chart III - Frequency Polygon

Chart III FREQUENCY POLYGON - RESULTS OF TEST

CLASS 8-143



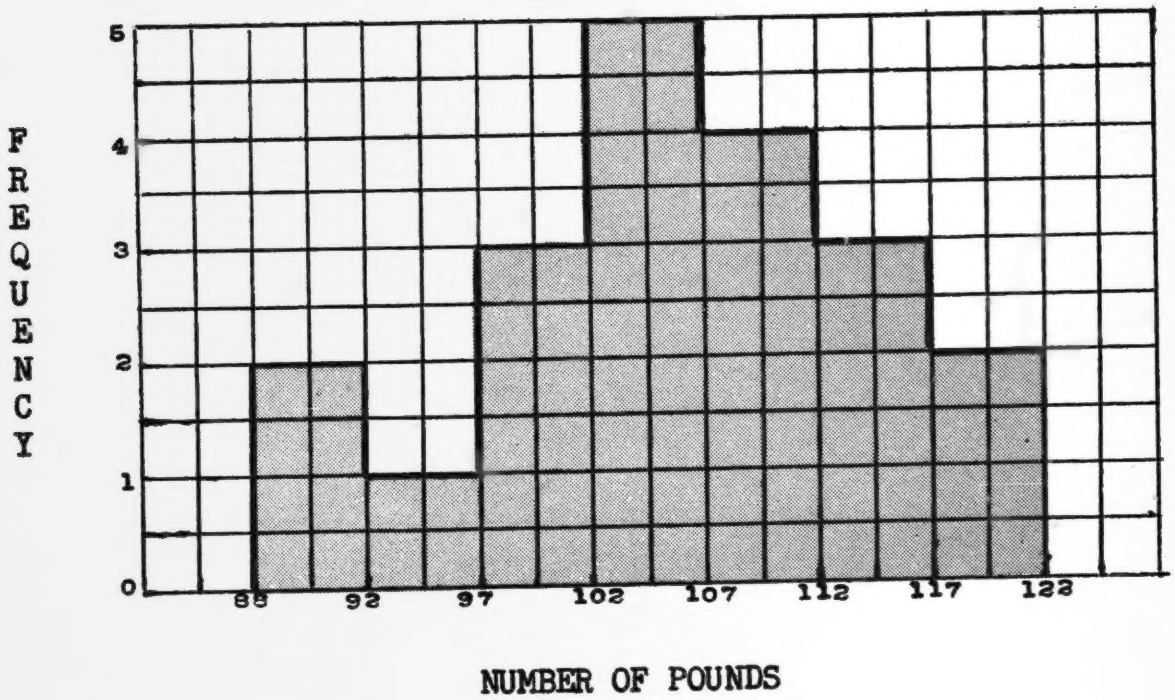
a. What is the title of the frequency polygon?
b. What is shown on the vertical scale? (frequency of the scores)
c. What is shown on the horizontal scale? (middle score of each interval)
d. Which score represents the interval with the greatest frequency?
e. How many pupils scored 100%?
f. How many pupils took the test?

II. Practice

A.

HISTOGRAM FOR FREQUENCY OF WEIGHTS

Class 8-143



1. What is the title of the histogram?
2. What is shown on the vertical scale? on the horizontal scale?
3. In which interval would a weight of 103 lbs. be recorded? (102-107)
4. Which interval has the greatest frequency?
5. How many pupils weigh about 75 pounds?

B. Explain how you would draw a frequency polygon from the above histogram.

C. Why do the line segments connecting the midpoints of the intervals have to begin and end on the horizontal axis in this type of graph?

D. Have pupils find other data, in current events, etc. that they can use to prepare charts similar to the above.

III. Summary

A. What is the advantage of grouping a large set of data into intervals before making a frequency table?

B. What type of graph does a histogram resemble?

C. What type of graph does a frequency polygon resemble?

D. What new vocabulary have you learned today?

Lesson 107

Topic: Statistics

Aim: To develop an understanding of the mean and the mode of a given set of data

Specific Objectives:

To develop the understanding of the mean
To learn the meaning of the mode

Challenge: In last year's Red Cross drive, class 8-333 made the following contributions:

10¢, 15¢, 15¢, 15¢, 5¢, 36¢, 13¢, 14¢, 16¢, 20¢,
13¢, 11¢, 10¢, 11¢, 10¢, 10¢, 15¢, 16¢, 20¢, 20¢,
15¢, 10¢, 15¢, 13¢, 10¢, 5¢, 5¢, 15¢, 20¢, 15¢, 15¢

What was the average contribution?

I. Procedure

A. Meaning and computation of the mean

1. Have pupils review through discussion the meaning of the following:

   Average monthly attendance for pupils in a class
   Average mark of a class in a test
   Average height of a group of pupils

2. Have pupils make a frequency distribution table of the data in the challenge and find the average contribution.

CONTRIBUTIONS TO RED CROSS BY CLASS 8-333

Contribution	Tally	Frequency	Total Contributions
5¢	///	3	15¢
10¢	~~////~~ /	6	60¢
11¢	//	2	22¢
13¢	///	3	39¢
14¢	/	1	14¢
15¢	~~////~~ ///	8	120¢
16¢	///	3	48¢
20¢	////	4	80¢
36¢	/	1	36¢
Total		31	434¢

Average $= \frac{434}{31} = 14$ Average contribution: 14¢

3. Does this mean that each pupil contributed 14¢? Explain.

4. Guide the pupils to an understanding of the meaning of average as a single number which represents the entire set of numbers. The average contribution of 14¢ implies that if all contributions had been the same, each contribution would have been 14¢. Tell pupils that another name for this average is the mean.

5. Show the pupils the effect of an extreme value on the mean. Use the data above but substitute a $5 contribution for one of the 10¢ contributions. Find the mean.

   How has the mean been affected by the single contribution of $5?

6. Elicit that since the value of each element is involved in computing the mean, an extreme value will affect the average.

7. Briefly discuss the effect on a student's average mark of one mark of zero.

B. Meaning and recognition of the mode

   1. Have pupils refer to table in A-2.

      a. Which score has the greatest frequency? (15¢)
      b. Tell pupils that the score which occurs most frequently in a set of scores is called the mode.
      c. Elicit that the mode is found by inspection; no computation is necessary.

II. Practice

A. John bowls on his high school team. In his last three-game series, he had the following scores: 180, 150, 160. What was his mean score? Suppose that in the first game his score had been 90. What would have been his mean score?

B. In a golf tournament, the 18-hold totals for the top nine players were: 69, 70, 71, 72, 73, 73, 73, 74, 74. Find the mean. Find the mode.

III. Summary

A. How do you find the mean of a set of numbers?
B. How do you recognize the mode in a set of data?
C. In using a single number to represent an entire set, the _____ is found by computation and the _____ is found by inspection.
D. What new vocabulary have we learned today?

Lesson 108

Topic: Statistics

Aim: To learn to compute the median

Specific Objectives:

To compute the median where there is an odd number of cases
To compute the median where there is an even number of cases

Challenge: The heights of 7 boys in a class are: 61", 57", 63", 60", 64", 58", 65". Which is the middle height?

I. Procedure

A. To find the median of an odd number of cases

1. Refer to challenge. Elicit that the heights should be ranked in order of size: 58 60 61 63 64 65 67

2. Which is the middle height? $58 \quad 60 \quad 61 \quad \overset{\downarrow}{63} \quad 64 \quad 65 \quad 67$

3. How many heights are below this middle height? How many above?

4. Tell pupils that this middle height is called the median. (The median is that score above and below which the number of scores is the same.)

5. If we have 31 ranked scores, which score will be the median? After several answers have been offered, suggest that the median score, in an odd number of cases, can be located by adding 1 to the total number of cases and dividing that sum by 2.

$$31 + 1 = 32$$
$$\frac{32}{2} = 16$$

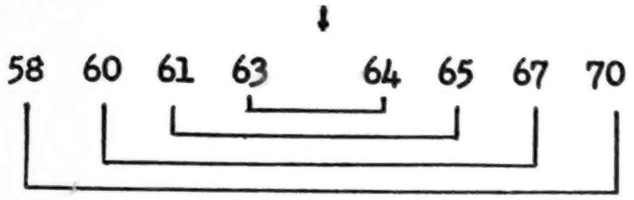
The 16th score is the median. Elicit that there will be 15 scores above and 15 scores below the median.

B. To find the median of an even number of cases

1. Have pupils note that in the case above, the middle height was found easily because there was an odd number of boys considered.

2. Suggest including the height of a boy 70" tall in the group. What is the median height?

3. Have pupils arrange the heights in order of size.
58 60 61 63 64 65 67 70
Let pupils pair off the scores as shown in the diagram.



4. Tell them since there is no middle score, we find the score halfway between 63 and 64. What is that score? (63½)

5. How many heights are less than 63½? How many greater? Does 63½ satisfy the meaning of median as expressed in A-4?

6. If we have 32 ranked scores, which score will be the median? After several answers have been offered, suggest that the median score, in an even number of cases, can be located as follows:

   a. Divide total number of scores by 2. $\frac{32}{2} = 16$

   b. The median will lie halfway between the 16th and 17th scores.

II. Practice

A. The marks (expressed in per cent) for 15 pupils in a mathematics class were: 92, 90, 84, 88, 92, 96, 86, 80, 84, 88, 96, 78, 80, 76, 82. Find the median score.

B. The noon temperatures for the first seven days of June were: 73°, 83°, 79°, 78°, 85°, 78°, 85°. Find the median temperature.

C. John worked as a camp counselor for the last six years. His summer earnings were: $90, $125, $150, $150, $150, $300. What was his average (mean) summer earnings per year? How does it compare with the median of his earnings?

D. Four boys bowled the following scores: 156, 180, 172, 186. Find the mean and the median.

III. Summary

A. What is meant by the median of a set of numbers?

B. How would you find the median of a set of 9 scores?

C. How would you find the median of a set of 10 scores?

D. How is the median influenced by one very great or one very small value?

Lesson 109

Topic: Statistics

Aim: To understand the meaning of the terms percentile and quartile

Specific Objectives:

To learn the meaning of percentile rank
To find a percentile from a frequency table
To understand the meaning of the terms "upper quartile" and "lower quartile"

Challenge: John took a qualifying test for a national honor society. His score was 520. Do you think he will be one of those accepted?

I. Procedure

A. To learn the meaning of percentile rank

1. Refer to challenge. Through discussion, elicit that the score has meaning only when it is compared with the scores of all those taking the test. Elicit that this can be done only by ranking all scores and counting down from the top.

2. Suppose John ranked 80th in the group taking the test. Will this additional information help us answer the challenge? Elicit that this information is of little help unless we know how many took the test. If only 100 students took the test, John's ranking was low; if 1000 students took the test, he has a high rank.

3. Let us suppose that 1000 students took the test. How many students ranked below John? (920) What per cent of the students were ranked below John? ($\frac{920}{1000} = 92\%$)

4. Tell the pupils that the percentile rank is the per cent of scores below a certain score. Therefore, John's percentile rank in this group is 92. Or, we can say John is in the 92nd percentile.

5. If the honor society accepted all those above the 90th percentile, will John be accepted?

6. Discuss briefly with pupils the use of percentiles in stating results of examinations involving large numbers of students such as certain standardized tests and some national scholarship examinations.

B. Finding the percentile rank from a frequency table

1. Have pupils consider the following frequency table:

Score	Number of Students
84	1
83	1
82	3
81	5
80	15
79	13
78	12
77	25
76	9
75	6
74	5
73	3
72	2

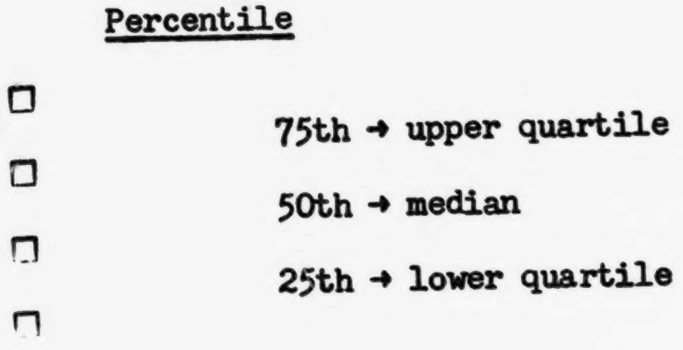
   a. How many students took the test?
   b. How many students fall below the score of 78?
   c. What per cent of the scores are below 78?
   d. What would be the percentile rank of a student who scored 78?
   e. What percentile rank would be given to a score of 82? (95th)

C. Meaning of upper quartile and lower quartile

1. In the table above, what percentile rank would be given to a score of 80? (75th)

2. What percentile rank would be given to a score of 77? (25th)

3. Tell the pupils that the 75th and 25th percentiles are respectively called the upper quartile and lower quartile. Have students observe the relation between "quartile" and "one-quarter."



4. Have pupils consider the following table:

Score	Number of Students
90	40
85	60
80	80
75	120
70	80
65	20

a. What is the total number of students?
b. Find the score which marks the 75th percentile. (85)
c. What is another name for the 75th percentile? (upper quartile)
d. Find the score which marks the 25th percentile. (75)
e. What is another name for the 25th percentile? (lower quartile)

II. Practice

A. Joan's score on a national achievement test had a percentile rank of 92. What does this statement mean? (92% of the students who took the test ranked below Joan.)

B. One university admits only high school graduates above the 80th percentile of their ranking in the class; another admits those above the 60th percentile. Of the students in this class, how many more would be admitted to the second university than to the first?

Score	Number of Students
90	20
85	20
80	40
75	60
70	40
65	20

C. In an examination, the upper quartile of the scores of 420 high school students was 80. What does this statement tell you about the set of students? (75%, or 315, of the students had less than 80.)

III. Summary

A. What is the meaning of percentile rank?
B. What is the meaning of upper quartile, lower quartile?
C. Name some situations where percentile rank would be used?
D. Discuss when a percentile rank would be more meaningful than a per cent score on a particular test.

Lesson 110

Topic: Probability

Aim: To introduce some of the concepts of probability

Specific Objectives:

To learn the meaning of probability
To use numbers to represent probability

Challenge: Each pupil in a class wrote his name on a separate piece of paper and then all the slips were placed into a box. If one slip is drawn out, what is certain about the name on the slip?

I. Procedure

A. Meaning of probability

1. Refer to challenge. Does the box contain the name of every pupil present? Is every name in the box the name of a member of the class? Have pupils answer the challenge. Elicit that the name on the slip is certain to be the name of one of the members of the class.

2. Have pupils give examples of names that are certain not to be on slips in the box, e.g., the principal's name, the name of a pupil in another class, etc.

3. John Smith's name is on one of the slips in the box. Only one slip is drawn from the box.

   Is it certain that John Smith's name is on the slip drawn?
   Is it certain that John Smith's name is not on the slip drawn?
   Is it possible that John Smith's name is on the slip drawn?

4. Tell pupils that a branch of mathematics which deals with the use of numbers to tell the likelihood of an event taking place is called Probability.

B. Using numbers to indicate the probability of an event

1. In a box there are 3 white balls. The first ball has only the letter X printed on it, the second ball only the letter Y, the third ball only the letter Z.

   a. If one ball is selected from the box, what is the probability that the ball will have only the letter T on it?

b. Tell pupils that since there is no chance that the ball selected has the letter T on it, the number zero is used to indicate its probability. In symbols this is written $P(E)=0$, which is read: "the probability of this event is zero."

2. If one ball is selected from the box, what is the probability that the ball will have a letter on it?

   a. Tell pupils that since this event (a selected ball has a letter on it) is certain to happen, the number 1 is used to indicate its probability. In symbols this is written $P(E)=1$, which is read: "the probability of this event is one."

   b. Have students summarize that if $P(E)=0$, it means an impossible event, and if $P(E)=1$, a certain event.

3. If one ball is selected from the box, what is the probability that the ball drawn will have the letter Y on it?

   a. Elicit that the likelihood of this happening is not impossible, nor is it certain. Therefore, the number used to indicate its probability must be greater than 0 and less than 1.

   b. Through questioning, have pupils realize that it is equally likely that any one of the three balls might be selected. Therefore, the chances that a ball with the letter Y on it will be selected is "one out of three" equally likely outcomes. The number $\frac{1}{3}$ is used to indicate the probability of this event. This is represented in symbols as $P(E)=\frac{1}{3}$.

II. Practice

A. The weatherman said it will probably rain tomorrow.

Is it certain to rain?
Is it certain not to rain?
Is it more likely to rain than not to rain?
Should you plan to go to the beach?

B. Using the symbols $P(E)=1$, $P(E)=0$, $0 < P(E) < 1$, indicate which of these events are certain to happen, cannot happen, may happen.

1. The sun will rise tomorrow.
2. It will rain tomorrow in New York City.
3. The Empire State Building will fly away.
4. Joe Brown will be an all American fullback.

C. We say the chances of a head coming up when a fair coin is tossed is "one out of two." In similar fashion, indicate the chances of

1. Drawing the king of hearts from a regular deck of 52 cards in a single drawing.

2. Guessing the right date if you know a person's birthday is in September.

3. Drawing a white ball from a bowl containing one white ball, one blue ball, and one green ball on the first drawing.

4. Throwing a "two" in a single toss with an ordinary die.

III. Summary

A. With what does the branch of mathematics called Probability deal?

B. What is the meaning of

$P(E) = 1$
$P(E) = 0$

C. $P(E) = \frac{1}{5}$ means that the probability of an event happening is

1. one chance out of five?
OR
2. five chances out of one?

Lessons 111 and 112

Topic: Probability

Aim: To develop the concept of the probability ratio

Specific Objectives:

To learn the meaning of the probability ratio
To find the probability ratio of chance events in problems

Challenge: A football captain called "heads" in each of the first five games and lost each toss. What should he call for the sixth game?

I. Procedure

A. Meaning of the probability ratio

1. Refer to challenge.

a. If a coin is tossed fairly (and we eliminate the possibility of standing on edge), in how many ways can it possibly turn up?

The way in which the coin may turn up is called an outcome or event.

The set of all possible outcomes may be represented as: $C = \{H,T\}$. One possibility is that the coin turns up head. $\{H\}$ is a subset of the set of all possible outcomes $\{H,T\}$. An event is a subset of the set of all outcomes.

b. What is the probability of a head turning up? (one out of two)

What ratio can be used to express this probability? ($\frac{1}{2}$)

Tell pupils that this ratio is called the probability ratio of a "head" turning up on the toss of a coin.

c. What is the probability ratio of a "tail" turning up? ($\frac{1}{2}$)

d. Elicit that in the challenge problem, since each toss is independent of the previous toss, no matter how many "heads" have turned up, the next toss is just as likely to be heads as tails. Since the outcomes are equally likely and no other possibility exists, the captain may call heads or tails with equal chance of success.

2. If we throw a die, what is the probability of a 3 (3 dots) turning up?

   a. How many faces has a single die? (6)

   b. What is the set of all possible ways that a die can turn up? D = {1,2,3,4,5,6} In how many ways can a 3 turn up?

   c. What is the probability of a 3 turning up? (one out of six)

   d. What is the probability ratio of a 3 turning up? Lead pupils to see that turning up a 3 is the desired outcome and is therefore considered a successful outcome. The probability ratio may therefore be expressed as

   $$P = \frac{\text{number of successful outcomes}}{\text{number of elements in the set of possible outcomes}}.$$

   Therefore, $P = \frac{1}{6}$.

B. Using the probability ratio

1. A red marble, a green marble, and a blue marble are put into a box.

   a. If, without looking, we were to pick one marble from the box, what is the set of all possible choices?
   M = {R, G, B}

   b. What is the probability ratio of picking a green marble on the first try?
   How many successful outcomes are there? (1)
   How many elements in the set? (3) Using

   $$P = \frac{\text{number of successful outcomes}}{\text{number of elements in the set of possible outcomes}}$$

   express the probability ratio. ($\frac{1}{3}$)

   c. What is the probability ratio of picking either a red or a green marble on the first try?
   How many successful outcomes are there? (2)
   How many elements in the set of possible outcomes? (3)
   Express the probability ratio. ($\frac{2}{3}$)

   d. What is the probability ratio of picking either a red or a green or a blue marble on the first try?
   How many successful outcomes are there? (3)
   How many elements in the set? (3)
   Express the probability ratio. ($\frac{3}{3}$ or 1)

2. The six faces of a child's block are lettered A,B,C,D,E,F.

   a. What is the probability that, if rolled, it will not turn up D?
      List the successful outcomes in this situation. (A,B,C,E,F)
      How many elements in the set?
      Express the probability ratio. ($\frac{5}{6}$)

   b. What is the probability that neither A nor D will turn up?
      Elicit that the successful outcomes in this situation are B,C,E,F.
      Express the probability ratio. ($\frac{4}{6}$ or $\frac{2}{3}$)

II. Practice

A. Using a new regular deck of fifty-two cards, what is the probability of picking on the first try

   the ace of spades?
   any ace?
   any spade?

B. A bag contains 3 white marbles, 4 blue marbles, 5 black marbles. Without looking, Morris selects a marble. What is the probability that the marble he selects is a black marble. What is the probability that the marble he selects is not blue?

C. The numbers painted on a "wheel of fortune" run from 1 to 20. After the wheel is spun, what is the probability

   that it will stop on 15?
   that it will stop on a number greater than 15?

III. Summary

A. What new vocabulary have you learned today?

B. In the probability ratio, what does the numerator represent? What does the denominator represent?

C. State some situations in which the probability ratio would be $\frac{1}{2}$, $\frac{1}{4}$, $\frac{2}{3}$.

Lessons 113 and 114

Topic: Probability

Aim: To develop the concept of a sample space

Specific Objectives:

To learn the meaning of a sample space
To use a sample space to find the probability ratio

Challenge: A penny and a nickel are tossed at the same time. What is the probability that both will turn up "heads"?

I. Procedure

A. Refer to challenge.

1. What are all the possible outcomes of tossing a penny and a nickel at the same time?

   Elicit that

   a. both coins could turn up heads, HH;
   b. the nickel could turn up heads and the penny could turn up tails, HT;
   c. the nickel could turn up tails and the penny could turn up heads, TH;
   d. both coins could turn up tails, TT.

   Have pupils organize this information in the form of a table as follows:

Nickel	Penny	Outcomes
H	H	H,H
H	T	H,T
T	H	T,H
T	T	T,T

2. Have pupils represent the set of all possible outcomes.
   S = {HH, HT, TH, TT}

3. Tell pupils the set {HH, HT, TH, TT} is the sample space for the experiment in the challenge. Each element in the sample space may be called a sample event or sample point. An event is a subset of the sample space.

4. What is the difference between HT and TH in paragraph 3?

5. Which subset of the sample space shows

   a head on each coin? ($\{HH\}$)
   a tail on each coin? ($\{TT\}$
   a head on the nickel and a tail on the penny? ($\{HT\}$)
   a head on one coin and a tail on the other? ($\{HT, TH\}$)

B. Using sample spaces to find probability ratios

1. Have pupils recall the generalization for the probability ratio.

$$P = \frac{\text{number of successful outcomes}}{\text{number of elements in the set of all possible outcomes}}$$

2. In the challenge problem,

   a. how many elements are in the sample space? (4)
   b. how many elements represent the successful outcome of both coins turning up heads? (1)
   c. what is the probability that both coins will turn up heads? ($\frac{1}{4}$)

3. If three coins, a penny, a nickel, and a dime are tossed at the same time, in how many different ways could they turn up?

   a. Help pupils construct a table showing the outcomes.

Nickel	Penny	Dime	Outcomes
H	H	H	H,H,H
H	H	T	H,H,T
H	T	H	H,T,H
H	T	T	H,T,T
T	T	T	.
T	H	T	.
T	T	H	.
T	H	H	.

   b. Have pupils complete the sample space when three coins are tossed. $S = \{HHH, HHT, HTH, HTT, \ldots\}$

   c. How many elements does the sample space contain? (8)

4. When tossing these three coins, what is the probability

   a. that all coins will turn up heads? ($\frac{1}{8}$)
   b. that the nickel and the penny will turn up heads? ($\frac{2}{8}$)
   c. that the dime will turn up heads? ($\frac{4}{8}$)
   d. that no coin will turn up heads? ($\frac{1}{8}$)

II. Practice

A. What is the probability that in a family of two children the first child will be a boy?

B. Two red balls and two green balls are placed in a bag.

1. If two balls are picked from the bag without looking, what is the probability that they will both be green?

2. What is the probability that one will be red and one will be green?

C. Suppose a nickel, a dime, and a quarter are tossed at the same time. What is the probability that at least two heads will turn up?

III. Summary

A. What is a sample space?

B. In a problem in probability, what does the expression "successful outcome" mean?

C. Explain how the construction of a sample space helps to find the probability of an event taking place.

Lesson 115

Topic: Probability

Aim: To learn to represent probability as a per cent

Specific Objectives:

To review expressing a ratio as a per cent
To learn to use per cent to represent the probability of an event happening

Challenge: The radio announcer reported that the probability of rain is 150%. Does this mean that

there will be a very heavy rainfall?
it might snow?
the radio announcer made a mistake?

I. Procedure

A. Expressing a ratio as a per cent

1. Have pupils recall that a ratio may be renamed as a per cent.

2. Rename the following as per cents.

$\frac{40}{100}$ $\frac{7}{10}$ $\frac{2}{5}$ $\frac{1}{4}$ $\frac{3}{2}$ $\frac{1}{3}$ $\frac{1}{6}$

B. Expressing probability as a per cent

1. Have pupils recall that when a coin is tossed fairly the probability of its turning up heads is $\frac{1}{2}$. How could this probability be expressed as a per cent?

2. What per cent chance is there that the coin will turn up tails?

3. What is the sum of the per cent found in example 1 and the per cent found in example 2?

4. What is the meaning of 100% probability? (certain to happen) Can you have more than 100% probability?

5. Refer to challenge. Elicit that 150% probability is meaningless and therefore the radio announcer made a mistake.

6. Lead pupils to realize that while probability ratios may be used to predict a successful outcome, often the event does not occur as predicted. The radio announcer said, "There is a 70% chance of rain."

a. Can you be sure that it will rain?

b. Is it possible that it will not rain?

c. Would it be wise to wear your best suit?

7. In a drawing for a prize, 100 slips were placed in a box.

   a. If a boy had his name on 10 of the slips, what is the per cent chance of his winning the prize?

   b. Could a girl win whose name is on only one slip?

   c. To be certain of winning, on how many slips would your name have to appear?

II. Practice

A. Express as per cents the following probabilities:

$\frac{3}{5}$ $\frac{5}{6}$ $\frac{1}{2}$ $\frac{81}{100}$

B. Complete the following:

1. A probability ratio is never greater than _____%.
2. If an event is certain to happen, the probability it will happen is _____%.
3. If two events are equally likely to happen, the probability of one of them happening is _____%.
4. In rolling a die, the chance of a 6 turning up is _____%.
5. If there are three white balls in a box, the probability of drawing a black ball is _____%.

III. Summary

A. How may a ratio be renamed as a per cent?

B. Discuss the meaning of a forecast of 70% chance of rain.

C. How would zero probability be expressed as a per cent?

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